## *Riemannian 4-symmetric spaces*

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RIEMANNIAN 4-SYMMETRIC SPACES

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ABSTRACT

This thesis studies the theory of Riemannian 4-symmetric spaces. It follows the methods first introduced by E. Cartan to study ordinary symmetric spaces, and extended by J. Wolf and A. Gray and by Kac.

The theory of generalized n-symmetric spaces was initiated by A. Ledger in 1967, and 2- and 3-symmetric spaces have already been classified. The theory of 4-symmetric spaces is completely new.

The thesis naturally divides into two chapters. The first chapter treats the geometry of the spaces. Their homogeneous structure and their invariant connections are studied. The existence of a canonical invariant almost product structure is pointed out. A fibration over 2-symmetric spaces with 2-symmetric fibers is obtained. Root systems are used to obtain geometric invariants. Finally a local characterization in terms of curvature is obtained.

Chapter II centers on the problems of classification. A local classification is given for the compact spaces in terms of simple Lie algebras. A global formulation is given for the compact classical simple Lie algebras.

A final section is devoted to invariant almost complex structures. A characterization is given in terms of their homogeneous structure. It is shown that they can bear both Hodge and non-Kähler structures.

# DEDICATION

A mi Esthercita hermosa y a mi jefazo,

con todo mi cariño y todo mi agradecimiento.

# RIEMANNIAN 4-SYMMETRIC SPACES

BY

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## INTRODUCTION

The theory of Riemannian 2-symmetric spaces was initiated by É. Cartan in 1926. Ever since, this theory has become a central subject in differential geometry. One of the most important features of this theory is its connection with the theory of semi-simple Lie groups. This connection is a source of very detailed and extensive information about the spaces. It was through it that E. Cartan accomplished their classification.

Owing to the importance of 2-symmetric spaces and to the wealth of their geometric structure, it is interesting to search for spaces whose theory provides in a natural way an extension of their theory.

In 1967, A. J. Ledger [29] initiated the study of generalized Riemannian symmetric spaces (see also P. J. Graham and A. J. Ledger [30]). He showed (cf. [16]) that on a Riemannian manifold $(M,g)$, the existence at each point $p$ in $M$ of an isometry with $p$ as an isolated fixed point was sufficient to ensure that $M$ was a Riemannian homogeneous space. Hence, the definition of Riemannian n-symmetric spaces is rather natural. Furthermore, the main property of the Cartan spaces is preserved. That is, the spaces are homogeneous manifolds of a well defined type (cf. Section 2), and therefore their theory is related to the theory of Lie groups. Hermitian 2-symmetric spaces provide a large class of examples which are n-symmetric for any $n$. On the other hand, O. Kowalski [14] showed the existence of generalized n-symmetric spaces (of arbitrary order $n$) which are not $m$-symmetric for $m < n$. Also, he has classified these spaces in low dimensions (dimension $\leq 5$).

Following Cartan's methods, J. Wolf and A. Gray gave in [28] the general structure theory for finite order inner automorphisms of the compact semi-simple Lie algebras. Then they accomplished a complete classification of the 3-symmetric spaces. These spaces are Riemannian manifolds which have associated in a natural way an invariant almost complex structure. A. Gray [8] showed that they are nearly Kähler manifolds and obtained a characterization of these spaces in terms of their curvature tensor.

The purpose of this thesis is to study the theory of Riemannian 4-symmetric spaces and to obtain their classification (for the compact simply connected 4-symmetric spaces). The coverage of the thesis is best explained by the following description of its contents.

The thesis naturally divides into two chapters. The first one is mostly concerned with the Riemannian geometry of the spaces, whereas the second chapter centers on the problems of classification. An additional last section is devoted to the study of invariant almost complex structures on these spaces.

Sections 1-3 are rather general and all the theory contained in them goes over (regular) $n$-symmetric spaces. Special emphasis is put on the important role of the regularity condition. This condition implies that the spaces can be represented as homogeneous manifolds of the form $G/K$, where $G$ is a connected Lie group with an automorphism of order four whose fixed point set is (essentially) $K$. The existence of a unique invariant connection on the principle bundle $G(G/K,K)$, invariant also under the symmetries, is proved. This connection yields a (local) characterization of the spaces. These results are (implicitly) contained in [16]. Although the proofs here are somewhat easier.

In Section 4 we point out a geometric feature that is intrinsic to 4-symmetric spaces. These spaces have canonically associated an almost product structure invariant under the symmetries.

$$TM = \mathcal{V} \oplus \mathcal{H} \quad \text{(orthogonal decomposition)}.$$

One of these distributions is integrable - we call it the vertical distribution $\mathcal{V}$. It is shown that it defines a regular foliation and that $M$ admits the (locally trivial) fibration

$$(*) \qquad \begin{array}{ccc} F & \hookrightarrow & M \\ & & \downarrow \\ & & B \end{array}$$

with $B$ affine 2-symmetric and $F$ Cartan symmetric. We study the Riemannian geometry of these fibrations. From our definition of 4-symmetric spaces it is clear that they are reflection spaces, and also that the above fibrations correspond to the fibrations obtained in [17]. However our proofs are straightforward, and no allusion is made to the theory of reflection spaces. In his work, [17] O. Loos did not make any reference to any Riemannian metric. Thus all the Riemannian geometry of these fibrations is new and may serve as a model to study the Riemannian geometry of reflection spaces and more generally the geometry of Riemannian submersions.

In Section 5 we study the geodesics of $M$ under the assumption that the fibration (*) is a Riemannian submersion. The main application of this is a description of the focal points of the fiber $F$ in terms of root systems.

In a different vein, in Section 6, we set about finding a local characterization in terms of curvature of Riemannian 4-symmetric spaces. Here the motivation comes from various sources. On one hand, Riemannian (locally) 2-symmetric spaces are characterized by having parallel curvature. Then, Kähler manifolds have parallel almost complex structures. Also, Riemannian 3-symmetric spaces are nearly Kähler manifolds for which the curvature tensor satisfies $(\nabla_X R)_{XJXXJX} = 0$ for all vector fields $X$ (see [8]). Thus we pose the problem of finding a characterization in the "same spirit" for Riemannian 4-symmetric spaces. We obtain one in terms of the symmetry tensor $S$ (the analog of an almost complex structure), the fundamental form of the distribution $\mathcal{H}$, and the curvature of $M$.

Chapter II begins with a small section on the de Rham decomposition of Riemannian 4-symmetric spaces. We show that the corresponding group of symmetries also decomposes.

In Section 8 we classify the automorphisms of order four of the compact semi-simple Lie algebras. The section is naturally divided into two parts. Part (a) treats the case of inner automorphisms. The classification here is carried along the same lines as in [28]. This method is very suitable for our purposes in Section 9.

We obtain the fixed point sets of the automorphisms and of the squares of the automorphisms, and give the corresponding tables for all the Lie algebras. A description of the base space B and the universal cover of the fiber F in (*) is given for each one of the entries in the tables. A global formulation is accomplished for the "classical" Lie algebras.* Part (b) follows the same pattern as part(a), but now for the outer automorphisms. Here we follow Kac (see Helgason [9] Ch.X).

Section 9 is concerned with almost complex structures on Riemannian 4-symmetric spaces invariant under the symmetries. It is shown (with one possible exception) that the compact simply connected almost Hermitian 4-symmetric spaces have non-vanishing Euler characteristic. These spaces are characterized as 4-symmetric spaces with a standard representation of the form G/K with K the centralizer of a torus. Every Hermitian 2-symmetric space of the compact type is one of these spaces. We also show that compact 3-symmetric spaces with a similar standard representation are also almost Hermitian 4-symmetric. It is shown that almost Hermitian 4-symmetric spaces (which are not Hermitian 2-symmetric) can have invariant Hodge

* We show that the important idea of duality in Cartan spaces can naturally be extended to 4-symmetric spaces. We also obtain the duals for the "classical" Lie algebras.

structures - hence in particular they are algebraic - and can also have invariant non-Kählerian structures (as opposed to the Cartan spaces).

# CHAPTER 1

## RIEMANNIAN GEOMETRY OF RIEMANNIAN 4-SYMMETRIC SPACES

### §1. Definition of Riemannian 4-Symmetric Spaces

In this section we start with the definition of Riemannian 4-symmetric spaces. They are Riemannian manifolds endowed at each point with a symmetry of order four that preserves the metric. Although the definition of these spaces arises as a natural way of extending the definition of Cartan (symmetric) spaces, their study can be fairly more complicated if an additional restriction is not imposed on them. Following Ledger [ 7 ] we include in our formal definition a regularity condition on the symmetries. Geometrically, this is an invariance condition which says that the set of symmetries is invariant under conjugation by the symmetries themselves. The consequences of this condition are more conveniently expressed when looking at them from the point of view of group theory. Such spaces have a transitive group of isometries. A result which is independent of whether or not the regularity condition is imposed. However, the regularity implies that the spaces can be represented as the homogeneous manifolds $G/K$ where $G$ is a connected Lie group with an automorphism of order four whose fixed point set is (essentially) $K$. This fact in turn will make feasible the classification of these spaces.

A Riemannian 4-symmetric space is a connected $C^\infty$-Riemannian manifold $(M,g)$ together with a family of isometries $(s_x)$ ($x$ in $M$), with the following properties:

(i) For each $x$ in $M$, the isometry $s_x$ is of order four and has $x$ as an isolated fixed point. $s_x$ will usually be called the symmetry at $x$.

(ii) (Regularity condition). For any two points $x$ and $y$ in $M$, the symmetries $s_x$ and $s_y$ satisfy

$$s_x \circ s_y = s_{s_x(y)} \circ s_x. \tag{1}$$

Comments. Clearly the above definition can be extended to symmetries of arbitrary order $n$, see for example [13]. For technical reasons it seems desirable to assume completeness in the definition, however, this will be a consequence of the fact that Riemannian 4-symmetric spaces are homogeneous Riemannian manifolds. Condition (ii) is not part of the original definition of n-symmetric spaces, but since we shall exclusively be concerned with the case when they are regular, we include it from the very beginning. The reason for this restriction is not an arbitrary one: There is a great difference between n-symmetric spaces and Cartan spaces, whereas that for Cartan symmetric spaces the condition that the symmetries be the geodesic involutions immediately guarantees their uniqueness, for n-symmetric spaces this uniqueness is missing. Thus, the map $s$ that assigns to each point $x$ in $M$ the symmetry $s_x$ can be rather arbitrary and complicated to work with, hence the imposition of such a restriction.

That regularity is the right sort of condition can be seen from different points of view. On one hand it is an invariance condition on the set of symmetries (conjugation by the symmetries themselves does not alter the set). On the other hand, it still allows us, up to some extent, to parallel the theory of Cartan spaces, especially when using Lie group theory and obtaining the classification for the compact spaces at the end of the present work. We prove here that this condition is satisfied by the Cartan spaces. Since we have to prove equality between two isometries, we only need to show that their differentials coincide at one point (and then use the exponential map and connectedness). We show that the differentials of $s_x \circ s_y$ and $s_{s_x(y)} \circ s_x$ coincide at $y$, $x$ and $y$ any two points in $M$,

$$(s_x \circ s_y)_{*_y} = (s_x)_{*_{s_y(y)}} \circ (s_y)_{*_y} = (s_x)_{*_y} \circ -I_{T_yM} = -(s_x)_{*_y} .$$

The first equality is merely the chain rule. The second equality is a consequence of the fact that $s_y$ is the geodesic involution at $y$, and therefore it has the negative of the identity map of $T_yM$ as its differential at $y$. Analogously we have:

$$(s_{s_x(y)} \circ s_x)_{*y} = (s_{s_x(y)})_{*s_x(y)} \circ (s_x)_{*y} = -I_{T_{s_x(y)}M} \circ (s_x)_{*y}$$

$$= -(s_x)_{*y}$$

as we wanted.

A similar argument can be used to give an alternate description of the regularity condition. For this, define the symmetry tensor $S$ as follows:

$S$ is the tensor field that associates to each point $x$ in $M$ the differential of the symmetry at $x$, i.e. $S_x = s_{x_*x}$. $S$ is a tensor field of type $(1,1)$ and preserves the metric. Furthermore, condition (i) in the definition implies that $S$ is of order four and has no eigenvalue $+1$. As done for the Cartan Spaces, we take the differential at $y$ of both sides in (1):

$$(s_x)_{*y} \circ (s_y)_{*y} = (s_{s_x(y)})_{*s_x(y)} \circ (s_x)_{*y}.$$

Using the symmetry tensor $S$, this can be written as

$$(2) \qquad (s_x)_{*y} \circ S_y = S_{s_x(y)} \circ (s_x)_{*y} \qquad (x,y \text{ in } M)$$

and as remarked above, by connectedness, the equality of the isometries in (1) is equivalent to the equality in (2). Therefore, the regularity condition (ii) is equivalent to the condition:

(ii)' The symmetry tensor $S$ is invariant under the action of the symmetries $(s_x)$ $(x$ in $M)$

We shall see that $S$ is not only differentiable, but analytic. Note that for the Cartan spaces, $S$ is minus the identity tensor, this gives an immediate explanation of the fact that they are always regular symmetric spaces. The tensor $S$ will play a prominent role in what follows.

In general, it will be said that a tensor field is s-invariant if it is invariant under the action of the symmetries $(s_x)$.

§2. Riemannian 4-Symmetric Spaces as Homogeneous Spaces

One of the central results in the theory of Riemannian n-symmetric spaces is that their groups of isometries act transitively on them.

Theorem [16]. Riemannian n-symmetric spaces are homogeneous Riemannian manifolds.

The idea of the proof is to show that if we take the closure in the full group of isometries of the subgroup generated by the symmetries $(s_x)$, then the orbit of one point under the action of this group is the whole manifold. Since the manifold is assumed connected and since this orbit is closed, it is only necessary to show that it is open. A thing that can be seen by using the symmetries.

An interesting consequence of the proof is that instead of having to work with the full group of isometries we can now restrict our attention to a group which is closely related to the symmetries and which is still large enough to act transitively and to carry information of the geometry that the symmetries induce on $M$. By connectedness, the identity component of this group also acts transitively. We denote it by $G$. It turns out that $G$ is well suited for our work. In fact, a complete characterization of Riemannian 4-symmetric spaces can be given purely in terms of $G$. See Proposition 1 below.

$M$ can be written as the homogeneous space $G/K$, with $K$ the isotropy group of $G$ at a point $0$ in $M$. $G$ acts effectively on $M$, and $K$ is compact.

The symmetries $(s_x)$ may or may not belong to $G$, nevertheless by conjugation they induce automorphisms $(\sigma_x)$ on $G$:

$$\sigma_x : G \to G, \quad \sigma_x(g) = s_x \circ g \circ s_x^{-1} \quad (g \in G.)$$

Each of these automorphisms is of order four, and for $\sigma_0$, from now on denoted by $\sigma$, the fixed point set $G^\sigma$ satisfies

$G_0^\sigma \subset K \subset G^\sigma$ where $G_0^\sigma$ is the identity component of $G^\sigma$.

This relation is a consequence of the regularity condition (1) in the definition. It can be seen as follows: since the symmetry tensor is invariant under the symmetries themselves, it is also invariant under $G$. In particular, if $k$ is any element in the isotropy group $K$, then $k_{*_0} \circ S_0 \circ k_{*_0}^{-1}$ equals $S_0$ hence $k \circ s_0 \circ k^{-1} = s_0$ for all $k$ in $K$. i.e. $k = s_0 \circ k \circ s_0^{-1}$ for all $k \in K$. This shows that $k$ belongs to the fixed point set $G^\sigma$. To see that $G_0^\sigma \subset K$, we take a one parameter group $\exp tX$, $t \in \mathbb{R}$, in $G_0^\sigma$ and show that it is contained in $K$, i.e. we show that $\exp tX \cdot 0 = 0$ for all $t$ in $\mathbb{R}$. Since $s_0$ has $0$ as an isolated fixed point, all that is necessary to prove is that the curve $\gamma(t) = \exp tX \cdot 0$ is left pointwise fixed by $s_0$:

$$s_0(\exp tX \cdot 0) = s_0(\exp tX \cdot s_0^{-1}(0)) = s_0 \circ \exp tX \circ s_0^{-1}(0) = (\exp \sigma tX) \cdot 0$$

and since $\exp tX$ belongs to $G_0^\sigma$, we have that this is equal to $\exp tX \cdot 0$ as we claimed.

We shall see later on, that conversely, the condition $K$ open in $G_0$ implies the regularity condition (1). (Proposition below in this section)

It is interesting to write the above results at Lie algebra level. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{k}$ the Lie algebra of $K$. Let $\sigma$ (same letter) be the automorphism on $\mathfrak{g}$ induced by $\sigma$. Then $\mathfrak{k}$ is the fixed point set of $\sigma$. Let $\upsilon$ be the eigenspace of $\sigma$ for the eigenvalue $-1$. Then

$\mathfrak{k} \oplus \nu$ is the eigenspace of $\sigma^2$ for the eigenvalue $+1$. Let $h$ be the eigenspace of $\sigma^2$ for the eigenvalue $-1$. Since $\sigma^2$ is an involution on $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \nu \oplus h$.

We claim that this decomposition is $Ad(K)$-invariant. In particular $G/K$ becomes a reductive homogeneous space:

Proof. Let $V \in \nu$ and $k \in K$, we show that $Ad(k)V \in \nu$ by showing that $\sigma(Ad(k)V) = -V$. But this is an easy calculation

$$\sigma(Ad(k)V) = Ad(\sigma k)\sigma V = Ad(k)\sigma V = -Ad(k)V.$$

Analogously if $H \in h$ then $\sigma^2(Ad(k)H) = Ad(\sigma^2 k)\sigma^2 H = -Ad(k)H \in h$.

For the sake of organization we summarize all the above results as the following:

Theorem. Let $(M,g,s)$ be a Riemannian 4-symmetric space. Let $G$ be the closure, in $I(M,g)$, of the subgroup generated by the symmetries $(s_x)$. Then

(i) $G$ acts transitively on $M$, and for a fixed point $0$ in $M$, $M$ can be written as the homogeneous space $G/K$ with $K$ the isotropy group of $G$ at $0$.

(ii) Conjugation with respect to $s_0$, the symmetry at $0$ induces an automorphism $\sigma$ of order four on $G$ such that the fixed point set $G^\sigma$ satisfies

$$G_0^\sigma \subset K \subset G^\sigma.$$

(iii) Let $\mathfrak{g}$ be the Lie algebra of $G$, and let $\sigma$ be the induced autormorphism by $\sigma$ on $\mathfrak{g}$. Then $\mathfrak{g}$ splits into the three vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \nu \oplus h$$

where $\mathfrak{k}$ is the Lie algebra of $K$, $\nu$ is the eigenspace of $\sigma$ for the eigenvalue $-1$ and $h$ is the eigenspace of $\sigma^2$ for the eigenvalue $-1$.

This decomposition is $\mathrm{Ad}(K)$-invariant and $G/K$ is a reductive homogeneous space. ///

The canonical projection $\pi$ from $G$ onto $G/K$ defines a principal bundle with structural group $K$. It will usually be denoted by $G(G/K,K)$. The differential $\pi_*$ at $e$ has kernel $= \mathfrak{k}$ and maps $\nu \oplus h$ isomorphically onto $T_K(G/K)$. This space in turn is isomorphic to $T_0M$. We shall always identify these spaces. As a consequence we have that any $G$-invariant structure on $M$ corresponds uniquely to an $\mathrm{Ad}(K)$-invariant structure on $\nu \oplus h$. Also, any $X \in \mathfrak{g}$ gives rise in a natural way to a Killing vector field on $M$. The one parameter group $\exp tX$ can be regarded as a one parameter group of isometries on $M$ thus defining a flow and hence a vector field $\tilde{X}$. By definition, $\tilde{X}$ is Killing. The correspondence $X \mapsto \tilde{X}$ from $\mathfrak{g}$ into $\chi(M)$ is a Lie algebra antihomomorphism.

The following proposition gives a characterization of homogeneous manifolds that are Riemannian 4-symmetric spaces.

Proposition. Let $G$ be a connected Lie group, and $\sigma : G \to G$ an automorphism of order four. Let $K$ be a subgroup with $G_0^\sigma \subset K \subset G^\sigma$, and write $\mathfrak{g} = \mathfrak{p} \oplus \nu \oplus \mathfrak{h}$ as in Theorem 1. If $\langle , \rangle$ is any inner product on $\nu \oplus \mathfrak{h}$ which is both $\mathrm{Ad}(K)$- and $\sigma$-invariant, then $\langle , \rangle$ induces a $G$-invariant metric on $G/K$ which makes $G/K$ into a Riemannian 4-symmetric space.

The proof is rather standard, the details can be found in [14]. We shall only mention how to construct the symmetries $(s_x)$, and show that the regularity condition is satisfied.

Construction of the symmetry $s_0$ at $0 = K \in G/K$: Since $K \subset G^\sigma$, $\sigma$ induces a diffeomorphism $s_0$ on $G/K$ that makes the following diagram commutative

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & G \\ \pi \downarrow & & \downarrow \pi \\ G/K & \xrightarrow{s_0} & G/K \end{array}$$

$s_0$ is the symmetry at $0$. (The condition $G_0^\sigma \subset K$ ensures that $0$ is an isolated fixed point). If now $x = g \cdot 0$ is any other point in $G/K$, $g$ in $G$. Then $s_x$ is defined by $s_x = g \circ s_0 \circ g^{-1}$. (Of course, it is necessary to show that it is well defined).

To prove the relation $s_x \circ s_y = s_z \circ s_x$, $z = s_x(y)$, we first establish the following identity

$$s_0 \circ g \circ s_0^{-1} = \sigma(g) \quad \text{or} \quad s_0 \circ g = \sigma(g) \circ s_0 \quad \text{for all } g \text{ in } G. \qquad (*)$$

(Note that here we are thinking of the elements of $G$ as acting on $G/K$, in general it will be clear from the context, how these elements are regarded).

Proof of (*). Let $\tilde{g}\cdot 0$ be any element in $G/K$, $\tilde{g}$ in $G$. Then $s_0 \circ g(\tilde{g}\cdot 0) = s_0(g\tilde{g}\cdot 0) = s_0(\pi(g\tilde{g})) = \pi\circ\sigma(g\tilde{g}) =$ $\sigma(g)\sigma(\tilde{g})\cdot 0 = \sigma(g)\cdot\pi\sigma(\tilde{g}) = \sigma(g)\circ s_0\circ\pi(\tilde{g}) = \sigma(g)\circ s_0(\tilde{g}\cdot 0)$.

///

Write $s_x = g\circ s_0\circ g^{-1}$, $s_y = g'\circ s_0\circ (g')^{-1}$, where $x = g\cdot 0$, $y = g'\cdot 0$, then $z = s_x(y) = g\circ s_0\circ g^{-1}(g'\cdot 0) =$ $g\circ s_0(g^{-1}g'\cdot 0) = g\circ\sigma(g^{-1}g')(s_0(0)) = g\sigma(g^{-1}g')\cdot 0$ thus $s_z = g\sigma(g^{-1}g')\circ s_0\circ\sigma((g')^{-1}g)g^{-1}$. We now compute $s_z\circ s_x$:

$$\begin{aligned} s_z\circ s_x &= (g\sigma(g^{-1}g')\circ s_0\circ\sigma((g')^{-1}g)g^{-1})\circ(g\circ s_0\circ g^{-1}) \\ &= g\circ s_0\circ(g^{-1}g')\circ s_0\circ((g')^{-1}g)g' \\ &= (g\circ s_0\circ g^{-1})\circ(g'\circ s_0\circ(g')^{-1}) = s_x\circ s_y. \end{aligned}$$

## §3. Invariant Connections on Riemannian 4-Symmetric spaces

In what follows we use some well known facts about invariant connections in homogeneous spaces. (See for example [12] Vol. II, Ch. X). $(M,g,s)$ is a Riemannian 4-symmetric space. As a homogeneous space is reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{p}\oplus\nu\oplus h$ given in (iii) above. Therefore on the principal bundle $G(G/K,K)$ there exists a canonical connection $\overline{\Gamma}$ (called in the literature the connection of the second kind with respect to the given splitting).

This connection is defined by the requirement that the horizontal distribution on $G$ be given by the right translates of $\mathfrak{m} = \mathfrak{v} \oplus \mathfrak{h}$. It is G-invariant, i.e., $G$ acts on $G/H$ as a group of affine transformations. What is more interesting is that the symmetries are also affine transformations with respect to $\overline{\Gamma}$, a result that follows immediately from the fact that the splitting of $\mathfrak{g}$ is $\sigma$-invariant. This connection is very special as can be seen from the following uniqueness result.

Proposition. On the principal bundle $G(G/K,K)$ there exists a unique G-invariant connection $\overline{\Gamma}$ which is also s-invariant. Furthermore, it coincides with the canonical connection with respect to the splitting of the Lie algebra $\mathfrak{g}$ as given above.

Covariant differentiation with respect to this connection will be denoted by $\overline{\nabla}$. The proof is rather standard. The existence part has already been established above. As for uniqueness, the idea is very simple, all that has to be done is to show that if a connection $\tilde{\Gamma}$ on $G(G/K,K)$ is both G-invariant and s-invariant, then the difference tensor defined by $D_X Y = \tilde{\nabla}_X Y - \nabla_X Y$, $X$, $Y$ in $\mathfrak{X}(M)$ is uniquely determined in terms of $\nabla$ and the symmetries $\{s_x\}$, i.e., $D$ is independent of the connection $\tilde{\nabla}$. Where $\nabla$ is the Riemannian connection.

Recall that every tensor that is G-invariant is parallel with respect to any connection on $G(G/K,K)$. In particular, since the symmetry tensor $S$ is G-invariant, $\tilde{\nabla} S \equiv 0$.

Hence, for any $X, Y$ in $\mathfrak{X}(M)$,

$$(\tilde{\nabla}_X S)Y - (\nabla_X S)Y = -(\nabla_X S)Y = (D_X S)Y = D_X SY - SD_X Y$$

and using the fact that both $\tilde{\nabla}$ and $\nabla$ are invariant under the symmetries it follows that $D$ is $S$-invariant, the term on the right hand side of the above equality can be written as $D_X SY - SD_X Y = D_X SY - D_{SX} SY = D_{(I-S)X} SY$. $S$ is nonsingular, of finite order and does not have eigenvalues $+1$, hence $I - S$ is nonsingular and $D$ has the form

$$D_X Y = -(\nabla_{(I-S)^{-1}X} S)S^{-1}Y \qquad X, Y \text{ in } \mathfrak{X}(M). \qquad ///$$

Note: This proposition compares with the similar result that in a Cartan symmetric space there exists a unique connection which is invariant under the involutions.

The connection $\bar{\nabla}$ has some interesting properties that justify its introduction: it is a metric connection, i.e. the Riemannian structure $g$ is parallel with respect to $\bar{\nabla}$ ($\bar{\nabla} g \equiv 0$). In fact, a more general result is true, and this is that any tensor field on $G/K$ which is $G$-invariant is parallel with respect to $\bar{\nabla}$. Hence, as $G$ acts as a group of affine transformations, $G$ leaves invariant $\bar{R}$ and $\bar{T}$, the curvature and torsion of $\bar{\nabla}$ respectively, it follows that $\bar{\nabla}\bar{R} \equiv 0$ and $\bar{\nabla}\bar{T} \equiv 0$. Also, we have already used the fact that $\bar{\nabla} S \equiv 0$.

However, there is a draw back of the connection $\overline{\nabla}$ with respect to the Riemannian connection $\nabla$, $\overline{\nabla}$ in general will not be torsion free, thus $\overline{T}(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] \neq 0$ for some $X, Y \in \mathfrak{X}(M)$. Actually, this is an essential difference between $\nabla$ and $\overline{\nabla}$, because if $\overline{T} \equiv 0$, then $\nabla = \overline{\nabla}$. As a consequence we would have that $\nabla R \equiv 0$ and hence that the space is Riemannian locally 2-symmetric. Note that $\overline{T} \equiv 0$ is equivalent to $\nabla S \equiv 0$. If $M$ is simply connected, then $M$ is a Riemannian (globally) 2-symmetric space (cf. [16], Section 5).

It is possible to give in terms of $\overline{\nabla}$ a local characterization of Riemannian 4-symmetric spaces purely tensorial.

<u>Proposition</u>. Let $(M,g)$ be a complete simply connected, connected, Riemannian manifold, and let $S$ be a tensor field of type $(1,1)$ of order $4$ with no eigenvalue $+1$.

Let $D_X Y = -(\nabla_{(I-S)^{-1}X} S)S^{-1}Y$ for $X, Y \in \mathfrak{X}(M)$

and $\overline{\nabla}_X Y = \nabla_X Y + D_X Y$.

Then $(M,g)$ is a Riemannian 4-symmetric space with symmetry tensor $S$ if and only if

(i) $g$, $\overline{T}$ and $\overline{R}$ are S-invariant, and

(ii) $\overline{\nabla} g = \overline{\nabla}\,\overline{T} = \overline{\nabla}\,\overline{R} = \overline{\nabla} S = 0$

Here $\overline{T}$ and $\overline{R}$ are the torsion and curvature of $\overline{\nabla}$ respectively.

Proof. This is only a slight reformulation of Theorem 4.11 in [15]. The local s-regularity in the condition (ii)(a) is here replaced by $\bar{\nabla} S = 0$. ///

## §4. The Almost Product Structure of A Riemannian 4-Symmetric Space

We have already seen that a Riemannian 4-symmetric space is a reductive homogeneous space $G/K$, with $\mathrm{Ad}(K)$-invariant decomposition of the Lie algebra of $G$ given by $\mathfrak{g} = \mathfrak{p} \oplus \nu \oplus h$. Here $\nu \oplus h$ is isomorphic to $T_0M$ under the canonical projection $\pi : G \to G/K$. It follows that $G/K$ has an induced almost product structure. The definition of this structure is rather standard, its consequences, however, are very remarkable. It turns out that one of the distributions defines a regular foliation, that the leaves of this foliation are Cartan symmetric spaces and that the space of leaves is in a natural way an affine 2-symmetric space. Furthermore, in the compact case, this fibration can be regarded as a Riemannian submersion with base a Cartan symmetric space.

Definition of the almost product structure. At each point $x \in M$, we define $V_x$ to be the subspace of $T_xM$ given by $V_x = g_{*_0}(\nu)$, where $g \in G$ is such that $g \cdot 0 = x$ and where $\nu \subset \mathfrak{g}$ has been identified with $\pi_*(\nu) \subset T_0M$, in particular, $V_0 = \pi_*(\nu)$. Analogously, $H_x$ is defined to be $g_{*_0}(h)$. Because of the $\mathrm{Ad}(K)$-invariance of the splitting of $\mathfrak{g}$, $V$ and $H$ are well defined and $G$-invariant. Also,

$T_xM = V_x \oplus H_x$ for all $x$ in $M$. Actually we can say more as the splitting of $\mathfrak{g}$ is $\sigma$-invariant, then $V$ and $H$ are s-invariant, i.e., invariant under the symmetries $(s_x)x$ in $M$. Also, since $v$ is the eigenspace of $\sigma$ for the eigenvalue $-1$ and $h$ is the eigenspace of $\sigma^2$ for the eigenvalue $-1$, $v$ and $h$ are orthogonal one to each other, and hence $V$ and $H$ are orthogonal complementary distributions, i.e., if $V$ and $H$ are vector fields with values in $V$ and $H$ respectively, then $g(V,H) \equiv 0$.

Remark. It is possible to describe $V$ and $H$ by means of the symmetry tensor. (Recall that the symmetry tensor is the tensor field of type (1,1) defined by $S_x = s_{x_*}x : T_xM \to T_xM$). At each point $x \in M$, $V_x$ is the eigenspace of $S_x^2$ for the eigenvalue 1, and $H_x$ is the eigenspace of $S_x^2$ for the eigenvalue $-1$. Observe that this characterization of $V$ and $H$ is true at 0 because $s_0 \circ \pi = \pi \circ \sigma$, and since $S$ is G-invariant it is true elsewhere.

The projections onto $V$ and $H$ are given by

$$pr_V = \frac{I+S^2}{2}, \qquad pr_H = \frac{I-S^2}{2}.$$

They will usually be denoted by $V$ and $H$ respectively (same letters). The distributions $V$ and $H$ do not share the same properties. On one hand we have that $H$ is always even dimensional, in fact, the restriction of $S$ to $H$ induces on it an almost complex structure. In general $H$

may or may not be integrable. On the other hand, $V$ is always integrable. Furthermore, it is autoparallel with respect to the Riemannian connection, thus defining a foliation on M with totally geodesic leaves.

In order to prove that $V$ is integrable, we use the homogeneous structure on M to explicitly define the integral submanifolds.

Let $G/K$ be the usual coset representation of M, and let $J \subset G$ be the fixed point set of $\sigma^2$. J is a closed subgroup and contains K. As K is compact, it then follows that $J/K$ is a closed submanifold of $G/K$. Let $F_0$ be the connected component of $J/K$ containing the point $K = 0$. Then $F_0$ is also closed in $G/K$ and can be written as the coset space $L/K$, where L is the (closed) subgroup of J that leaves $F_0$ invariant

<u>Proposition</u>. Let $F_0$ be as above. Then

(i) $F_0$ is an integral submanifold of $V$.

(ii) For each x in $F_0$, the symmetry $s_x$ leaves $F_0$ invariant.

(iii) $F_0$ is a complete totally geodesic submanifold.

<u>Proof</u>. We first prove (ii). We start by showing that it is only necessary to prove that $s_0$ preserves $F_0$: Let x be an arbitrary point in $F_0 = L/K$, $x = \ell K$ for some $\ell \in L$. Then the symmetry at x is given by $s_x = \ell \circ s_0 \circ \ell^{-1}$,

therefore $s_x(F_0) = \ell \circ s_0 \circ \ell^{-1}(F_0) = \ell \circ s_0 \circ \ell^{-1}(L/K) = \ell \circ s_0(L/K)$. It follows that $s_x$ preserves $F_0$ if and only if so does $s_0$. In order to prove that $s_0$ leaves invariant $F_0$, we show that $s_0(J/K) = J/K$, then by continuity and the fact that $s_0$ leaves $0$ fixed, the result will follow. But that $s_0$ preserves $J/K$ is an immediate consequence of the relation $\pi \circ \sigma = s_0 \circ \pi$ (since $\sigma(J) = J$ and $J/K = \pi(J) = \pi(\sigma J) = s_0 \circ \pi(J) = s_0(J/K)$). Furthermore, this relation, in terms of elements, tells us that $s_0(gK) = s_0(\pi g) = \pi\sigma(g) = \sigma(g)K$ for all $g \in G$. In particular $s_0^2(\ell K) = \sigma^2(\ell)K = \ell K$ for all $\ell \in L$, this is, $s_0^2 | F_0 = \mathrm{id}$. This shows that $F_0$ with the induced Riemannian metric and with the restrictions of the symmetries $(s_x)x \in F_0$ becomes a Cartan symmetric space. Note that $J/K$ is the fixed point set of $s_0^2$, hence it is a totally geodesic submanifold of $G/K$, in particular, since $F_0$ is open in $J/K$, $F_0$ is also a totally geodesic submanifold of $G/K$. This proves (iii). It is now easy to see that $F_0$ is an integral submanifold of $V$. As noted above, $J/K$ is the fixed point set of $s_0^2$, thus at $0$, $T_0(J/K) = T_0F_0$ is the fixed point set of $s_{0_{*_0}}$ which is $V_0$. At any other point $x = \ell K \in L/K$ we have $V_x = \ell_{*_0}(V_0) = \ell_*(T_0F_0) = T_xF_0$ as claimed. ///

An immediate consequence of (i) is that $V$ is an involutive distribution. This can be seen as follows: Let $x = gK$ be an arbitrary point of $G/K$, then because of the

G-invariance of $\mathcal{V}$, $gF_0$ is an integral submanifold of $\mathcal{V}$ through x. Furthermore, it is also totally geodesic and invariant under the symmetries $(s_y)$, $y \in gF_0$. Hence, we have proved the following theorem.

Theorem. The distribution $\mathcal{V}$ defines a foliation on M with leaves complete totally geodesic submanifolds. Furthermore, each leaf inherits in a natural way the structure of a Cartan symmetric space. ///

Up to this point we have looked at the structure of each one of the leaves separately. What is also an important result is that this foliation is in fact regular, that is, the space of leaves has a natural structure of differentiable manifold which makes the canonical projection into a differentiable submersion. All this is very straightforward, one only has to notice that L is a closed subgroup of G, and hence that the quotient space G/L is an analytic manifold. Thus we have the following locally trivial fibration

$$\begin{array}{ccc} L/K & \hookrightarrow & G/K \\ & & \downarrow \\ & & G/L \end{array}$$

(see for example [12] Vol. I Ch. 1). Note that L satisfies $G_o^{\sigma^2} \subset L \subset G^{\sigma^2}$, which says that G/L is an affine 2-symmetric space. Thus we have proved the

Theorem. Every Riemannian 4-symmetric space fibers over an affine 2-symmetric space with fibers the integral submanifolds of the distribution $V$. In particular, all the fibers are isometric to a Cartan symmetric space. ///

This suggests the adoption of the following terminology: $V$ and $H$ will be called the vertical and horizontal distributions respectively.

There is one interesting case that deserves special attention, this is when M is compact. Here we have that all the above groups are compact and hence G/L can be endowed with a G-invariant Riemannian metric that makes it into a Cartan symmetric space. Furthermore, the metrics on G/K and G/L can be chosen so as to make the projection a Riemannian submersion. Thus in the compact case, we may think of a Riemannian 4-symmetric space as the total space of a locally trivial fibration over a Riemannian 2-symmetric space with fiber a Riemannian 2-symmetric space and with projection a Riemannian submersion.

We shall study this type of fibrations in the following section.

## §5. Geodesics and Riemannian Submersions. Focal Points

The geodesics of the connection $\overline{\nabla}$ issuing from 0 have an explicit description in terms of the decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{v} \oplus h$ in §2, they are the orbits of 0 by the one parameter groups $\exp tX$ with $X$ in $\mathfrak{v} \oplus h$. In general, they do not coincide with the geodesics of the Riemannian connection. This is the condition for a

Riemannian homogeneous space to be naturally reductive (with respect to the given splitting). Cartan symmetric spaces are always naturally reductive. This is a consequence of the uniqueness of a connection invariant under the geodesic involutions. Thus $\nabla$ and $\overline{\nabla}$ are the same. A. Gray has shown that Riemannian 3-symmetric spaces are naturally reductive whenever the associated almost complex structure is nearly Kahlerian, see [ 8 ]. As for Riemannian 4-symmetric spaces I do not know whether or not they are naturally reductive with respect to the above splitting. However, for geodesics issuing from 0 with initial direction on either $\mathcal{v}$ or $\mathcal{h}$ a similar description as orbits of one parameter groups can be given.

The result for geodesics with initial direction on $\mathcal{v}$ is an immediate consequence of the Proposition in §4 and the similar result for Cartan symmetric spaces. As for geodesics with initial direction on $\mathcal{h}$, we prove

<u>Proposition</u>. Assume the projection $\pi : M \to B$ in §4 is a Riemannian submersion. Then for each $X \in \mathcal{h}$, the orbit of 0 under the one parameter group $\exp tX$ is a horizontal geodesic in M.

<u>Proof</u>. Let $\tilde{X}$ be the induced Killing vector field on M by X. It is necessary to show that $\nabla_{\tilde{X}}\tilde{X}$ vanishes identically along the curve $\gamma(t) = \exp tX \cdot 0$. We prove separately that both $V(\nabla_{\tilde{X}}\tilde{X})$ and $H(\nabla_{\tilde{X}}\tilde{X})$ vanish at 0. Then the

result will follow since $\widetilde{X}$ is Killing and $\exp tX \cdot 0$ is an integral curve.

At the origin $\widetilde{X}$ is horizontal, hence $\mathcal{V}(\nabla_{\widetilde{X}}\widetilde{X})_0 = A_{\widetilde{X}_0}\widetilde{X}_0$ in O'Neill's notation [20], and since $A_X Y = \frac{1}{2}\mathcal{V}[X,Y]$ for any pair $X$ and $Y$ of horizontal vector fields, $\mathcal{V}(\nabla_{\widetilde{X}}\widetilde{X})_0 = 0$.

To prove that $\mathcal{H}(\nabla_{\widetilde{X}}\widetilde{X})_0 = 0$ we only make use of the square of the symmetry $s_0$. Since $s_0^2$ is an isometry, $s_0^2(\nabla_{\widetilde{X}}\widetilde{X})$ is $s_0^2$-related to $\nabla_{s_0^2\widetilde{X}} s_0\widetilde{X}$, hence $(s_0^2(\nabla_{\widetilde{X}}\widetilde{X}))_0 = s_{0*}^2\ (\nabla_{\widetilde{X}}\widetilde{X})_0$ is equal to $(\nabla_{s_0^2\widetilde{X}} s_0^2\widetilde{X})_0$. By the tensor nature of $\nabla_{(\cdot)}X$, we can substitute $(s_0^2\widetilde{X})_0$ by $-\widetilde{X}_0$. Thus we have $s_{0*}^2(\nabla_{\widetilde{X}}\widetilde{X})_0 = -\nabla_{\widetilde{X}_0}(s_0^2\widetilde{X})$ and taking horizontal components: $-\mathcal{H}(\nabla_{\widetilde{X}}\widetilde{X})_0 = -\mathcal{H}(\nabla_{\widetilde{X}_0} s_0^2\widetilde{X})$. To conclude we show that $s_0^2\widetilde{X} = -\widetilde{X}$ along $\exp tX \cdot 0$.

Assume this for a moment. Then our last relation gives $-\mathcal{H}(\nabla_{\widetilde{X}}\widetilde{X})_0 = -\mathcal{H}(\nabla_{\widetilde{X}_0} s_0^2\widetilde{X}) = \mathcal{H}(\nabla_{\widetilde{X}_0}\widetilde{X}) = 0$ which is what we wanted to prove. Thus we only have to prove that $s_0^2\widetilde{X} = -\widetilde{X}$ along $\exp tX \cdot 0$. By definition, $(s_0^2\widetilde{X})\exp tX \cdot 0 = s_{0*}^2\ \widetilde{X}s_0^{-2}(\exp tX \cdot 0)$, now $\widetilde{X}$ at that point is the tangent at $0$ of the curve $\tau(s) = \exp sX \cdot s_0^{-2}(\exp tX \cdot 0)$ thus

$$(s_0^2\widetilde{X})\exp tX \cdot 0 = \frac{d}{ds}\bigg|_{s=0} s_0^2(\tau(s)) = \frac{d}{ds}\bigg|_{s=0}\{s_0^2 \exp sX\, s_0^{-2}\exp tX \cdot 0\}.$$

Note that $s_0^2 \exp tX\, s_0^{-2}(0) = \exp t\sigma^2 X \cdot 0 = \exp - tX \cdot 0$ is also equal to $s_0^2 \exp tX \cdot 0$ since $s_0^{-2}(0) = 0$. As $s_0^4 = \mathrm{id}$, we have $s_0^2 \exp sX\, s_0^{-2} s_0^2 s_0^2 \exp tX \cdot 0 = s_0^2 \exp sX \exp - tX \cdot 0 = s_0^2 \exp(s-t)X \cdot 0$ and using again the last remark this is equal to $\exp - (s-t)X \cdot 0$ which is $\exp - sX \cdot \exp tX \cdot 0$, and differentiating with respect to $s$ at the origin we get $-\tilde{X} \exp tX \cdot 0$ which is what we wanted. ///

Having described the horizontal geodesics issuing from $0$, it is now possible to describe the horizontal geodesics issuing from any other point $x$ on the fiber $F_0$. For this we use the fact that $F_0$ can be written as the homogeneous space $L/K$ with $L$ a group of isometries not only of $F_0$ but also of $M$. Then $x = \ell \cdot 0$ for some $\ell$ in $L$. Hence the horizontal geodesics issuing from $x$ are of the form $\gamma(t) = \ell(\exp tH \cdot 0)$ with $H$ in $h$. $\gamma(t)$ can be written as $\gamma(t) = \ell \exp tH\, \ell^{-1} \ell \cdot 0 = \exp \mathrm{Ad}(\ell) tH\, \ell \cdot 0 = \exp t\, \mathrm{Ad}(\ell) H \cdot x$. i.e., $\gamma(t)$ is the integral curve of the induced Killing vector field $\mathrm{Ad}(\ell)H$ that passes through $x$. Note that $h$ is $\mathrm{Ad}(L)$-invariant, hence $\mathrm{Ad}(\ell)H$ is an element $H'$ in $h$. This shows that the horizontal geodesics along the fiber $F_0$ are the integral curves of the induced Killing vector fields $\tilde{H}$ with $H$ in $h$. In particular, the normal bundle to $F_0$, $\nu(F_0)$, is parallelizable.

Using this characterization of the horizontal geodesics, it is possible to give a description, similar to the one given for conjugate points in a Cartan symmetric space, for focal points to the fibers in terms of roots of the Lie algebra of $\mathfrak{g}$.

For this, we let $M = G/K$ be as usual the coset representation of a Riemannian 4-symmetric space, but now with the additional assumption $G$ compact and semi-simple. We also assume the fibration in §4 to be a Riemannian submersion. Under these conditions, we describe the focal points of the fiber through 0. In doing so, we need to recall some results for conjugate points on Cartan symmetric spaces, see [9] Ch. VII, and quote a result by O'Neill in [21].

The Lie algebra $\mathfrak{g}$ of $G$ is compact and semisimple, the automorphism $\sigma$ of order 4 gives the splitting $\mathfrak{g} = \mathfrak{k} \oplus \nu \oplus h$ and if we set $\lambda = \mathfrak{k} \oplus \nu$, then $\mathfrak{g} = \lambda \oplus h$ is a symmetric Lie algebra of the compact type with involution $\sigma^2$ satisfying $\sigma^2|\lambda = \mathrm{id}$ and $\sigma^2|h = -\mathrm{id}$. The manifold $B = G/L$ is a Cartan symmetric space. $h$ is identified with the tangent space of $B$ at 0 and the Riemannian exponential map is the exponential map of $G$ restricted to $h$ followed by the canonical projection $\pi$ from $G$ onto $G/L$.

Let $\mathfrak{a}_h$ be a maximal abelian subspace of $h$ and let $\mathfrak{a}$ be a maximal abelain subalgebra of $\mathfrak{g}$ generated by $\mathfrak{a}_h$. Let $\mathfrak{h}$ be the complexification of $\mathfrak{a}$ in $\mathfrak{g}^{\mathbb{C}}$. Then $\mathfrak{h}$ is a Cartan subalgebra. The root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}$ is denoted by $\Delta$, and $\Delta_{\mathfrak{a}}$ denotes the set of roots in $\Delta$ which do not vanish identically on $\mathfrak{a}_h$. Then we have

Proposition. The point $X = \mathrm{Ad}(\ell)A$ is conjugate to $0$ in $B$ if and only if $\alpha(A) \in \pi\sqrt{-1}(\mathbb{Z}-0)$ for some $\alpha \in \Delta\mathfrak{a}_h$, $A$ is an element in $\mathfrak{a}$. (Note that any element is of the above form, since $h = \cup_{\ell \in L} \mathrm{Ad}(\ell)\mathfrak{a}_h$). We recall the following result by O'Neill, [21].

Theorem. Let $\pi : M \to B$ be a submersion, $\gamma : [a,b] \to M$ a horizontal geodesic. Then the following integers are equal:

1. The order of $\gamma(b)$ as a focal point of the fiber $F_a$ along $\gamma$.
2. The order of $\gamma(a)$ as a focal point of the fiber $F_b$ along $\gamma$.
3. The order of conjugacy of the end points of $\pi \circ \gamma$ along $\pi \circ \gamma$.

With this in mind we can state (same notation as above)

Proposition. Let $H \in h$. Then the point $\exp H \cdot 0$ in $M$ is a focal point of the fiber $F_0$ along the (horizontal) geodesic $\exp tH \cdot 0$ if and only if $\alpha(A) \in \pi\sqrt{-1}(\mathbb{Z}-0)$.

## §6. A Local Characterization of Riemannian 4-Symmetric Spaces

Riemannian 4-symmetric spaces in some sense resemble complex manifolds. Both of these classes of spaces have associated in a natural way a tensor field of type (1,1) of order four. In the case of complex manifolds, this tensor is the induced complex structure $J_x$ on each tangent space $T_xM$. In the case of a Riemannian 4-symmetric space, this tensor is the symmetry tensor $S$. The main difference between these two types of structures is that whereas $J$ never has eigenvalues $-1$, $S$ in general may admit $-1$ as an eigenvalue. In fact, in order to avoid falling into the realm of symmetric spaces it is convenient to assume from the outset that $-1$ is an eigenvalue of $S$.

This analogy provides us with a wealth of questions. Concretely one may try to parallel the theory of complex manifolds and hence one can always ask if a theorem in complex geometry has its counterpart for Riemannian 4-symmetric spaces.

A better understanding of complex manifolds is obtained if one first studies them from the more general setting of almost complex manifolds. Then, the first fundamental result is that an almost complex manifold with almost complex structure $J$ is a complex manifold with induced complex structure $J$ if and only if the torsion of $J$ vanishes identically. (The torsion of $J$ is defined to be $N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\}$ for $X, Y \in \mathfrak{X}(M)$). This result is purely of the complex differential geometry of $M$ and no allusion is made to any sort of Riemannian geometry. When a Riemannian metric is introduced so that it is compatible with the complex structure, then we have an Hermitian metric. In this setting, the basic result is provided by the characterization of Kähler manifolds: Let $(M,g,J)$ be a Hermitian manifold, then it is Kählerian if and only if $\nabla_X J \equiv 0$ for all $X \in \mathfrak{X}(M)$.

In [15] are investigated those Riemannian n-symmetric spaces $(M,g,s)$ for which the symmetry tensor $S$ is integrable, in the sense that its torsion vanishes. (The torsion is defined as $S^2[X,Y] - S[SX,Y] - S[X,SY] + [SX,SY]$.) It is proved that integrability of $S$ is equivalent to $S$ being parallel. Moreover, it turns out that the Riemannian manifold $(M,g)$ underlying $(M,g,s)$ is Riemannian 2-symmetric.

The question of integrability of $S$ being settled, we take a different approach to study Riemannian 4-symmetric spaces, our aim being to arrive at a local characterization of these spaces. For example, we have that Riemannian 2-symmetric spaces are characterized (locally) as those spaces for which its curvature tensor is parallel. Riemannian 3-symmetric spaces are characterized in terms of their associated almost complex structure $J$ (see [8]) as those spaces for which $(\nabla_X J)X = 0$ (nearly Kähler condition) and also $(\nabla_X R)_{XJX\,XJX} = 0$ for all $X \in \mathfrak{X}(M)$. Also, as we pointed out above, Kähler manifolds are defined to be Hermitian manifolds with $\nabla_X J \equiv 0$. In the light of all this, we pose the following problem:

(P) Given a tensor field $S$ of type $(1,1)$ of order four and with no eigenvalue $+1$ on a Riemannian manifold $(M,g)$, under what sort of geometric restrictions is $S$ the symmetry tensor of a family of symmetries?

There does exist an immediate answer. This is given by the theorem of Cartan-Ambrose: $S$ must preserve $g$ and all the covariant derivatives $\nabla^{(n)}R$ of the curvature tensor $R$. Of course this result is of local nature and we also have to check if the regularity condition of the induced isometries is satisfied. However, this is an infinite set of conditions. Thus we shall start by searching for a finite set of conditions. We would like this set to be geometrically appealing and easy to manipulate. We have already obtained a finite set of necessary and sufficient conditions in Section 3. There, it was stated that a Riemannian manifold $(M,g)$ - $M$ complete and simply connected - is a Riemannian 4-symmetric space with symmetry tensor $S$ if and only if

(i) $g, \overline{T}$ and $\overline{R}$ are $S$-invariant, and

(ii) $\overline{\nabla}g \equiv 0$, $\overline{\nabla}\overline{T} \equiv 0$, $\overline{\nabla}\overline{R} \equiv 0$, $\overline{\nabla}S \equiv 0$

the bars denote covariant differentiation, torsion and curvature with respect to the connection defined by

$$\overline{\nabla}_X Y = \nabla_X Y - (\nabla_{(I-S)^{-1}X} S)S^{-1}Y \qquad X,Y \in \mathfrak{X}(M) \qquad \text{(see §3).}$$

The idea now is: given a Riemannian manifold $(M,g)$ complete and simply connected with a symmetry tensor $S$ as above, state explicitly the above conditions (i) and (ii) in terms of the Riemannian connection. Our final results are given at the end of the section summarized in two theorems.

To begin with, we shall write the difference tensor $D_X Y = -(\nabla_{(I-S)^{-1}X} S)S^{-1}Y$ in a more appropriate way for our calculations. For this we use S-invariance and the induced almost product structure on M.

The S-invariance of D simply means $SD_X Y = D_{SX} SY$. Thus $SD_{VX} VY = D_{SVX} SVY = D_{VX} VY$, hence $VD_{VX} VY = 0$, and also using $S^2$ instead of S, $HD_{VX} VY = 0$, i.e., $D_{VX} VY = 0$ for all X and Y.

This is a relation that should be expected since the leaves of $V$ being totally geodesic Cartan symmetric spaces admit a unique invariant connection, thus $\nabla = \overline{\nabla}$ on the leaves and $D_{VX} VY = 0$.

Similarly we have

$$H(D_{HX}(HY)) = 0, \quad V(D_{VX}(HY)) = 0, \quad V(D_{HX}(VY)) = 0.$$

We omit the proofs.

From this we have that $D_X Y$ decomposes into three parts

$$D_X Y = VD_{HX} HY + HD_{HX} VY + HD_{VX} HY.$$

We associate the first two terms on the right to define the tensor $-A_X Y$. We shall see that this tensor is the same as the tensor A as defined in [20]. Since both $H$ and $V$ are invariant under the group of symmetries, $\overline{\nabla} H \equiv 0$ and $\overline{\nabla} V \equiv 0$. Now,

$$0 = \mathcal{V}(\overline{\nabla}_{\mathcal{H}X}\mathcal{H})\mathcal{H}Y = \mathcal{V}\overline{\nabla}_{\mathcal{H}X}\mathcal{H}Y - \mathcal{V}\mathcal{H}\overline{\nabla}_{\mathcal{H}X}\mathcal{H}Y = \mathcal{V}\nabla_{\mathcal{H}X}(\mathcal{H}Y) + \mathcal{V}D_{\mathcal{H}X}\mathcal{H}Y.$$

This shows two things, on one hand, that $\mathcal{H}$ is auto-parallel with respect to $\overline{\nabla}$, and on the other hand, that $\mathcal{V}D_{\mathcal{H}X}\mathcal{H}Y = -\mathcal{V}\nabla_{\mathcal{H}X}(\mathcal{H}Y)$.

Analogously, if we write down explicitly the relation $0 = \mathcal{H}(\overline{\nabla}_{\mathcal{H}X}\mathcal{V})\mathcal{V}Y$ we obtain $\mathcal{H}D_{\mathcal{H}X}\mathcal{V}Y = -\mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y$. Thus

$$D_XY = \mathcal{H}D_{\mathcal{V}X}\mathcal{H}Y - \mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y - \mathcal{V}\nabla_{\mathcal{H}X}\mathcal{H}Y = \mathcal{H}D_{\mathcal{V}X}\mathcal{H}Y - A_XY.$$

It is easy to check that $\mathcal{H}D_{\mathcal{V}X}\mathcal{H}Y = \frac{1}{2}\mathcal{H}(\nabla_{\mathcal{V}X}S)S\mathcal{H}Y = \frac{1}{2}(\nabla_{\mathcal{V}X}S)S\mathcal{H}Y$. The $\mathcal{H}$ can be suppressed because $\mathcal{V}$ is totally geodesic with respect to $\nabla$.

Finally we have: $D_XY = -A_XY + \frac{1}{2}(\nabla_{\mathcal{V}X}S)S\mathcal{H}Y$

$$\overline{\nabla}_XY = \nabla_XY - A_XY + \frac{1}{2}(\nabla_{\mathcal{V}X}S)S\mathcal{H}Y$$

for all $X, Y$ in $\chi(M)$ (1)

and recall that $A_XY = \mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{H}X}\mathcal{H}Y$.

The task we have now is: given the connection $\overline{\nabla}$ in (1), obtain a set of geometric restrictions which ensures that the relations in (i) and (ii) above are satisfied. The way we proceed is as follows: first we unwind the conditions in (i) and (ii) and then show that what we obtain is indeed a good set of restrictions.

In all that follows $X, Y, Z\ldots$ will denote smooth vector fields on $M$. Note that the $S$-invariance of $g$ can always be assumed without any loss of generality.

The condition $\bar{\nabla} g \equiv 0$

It is well known that for a connection with parallel torsion and parallel curvature, torsion and curvature preserving linear transformation can be realized by means of affine transformations (locally), the extra condition we have: $\bar{\nabla} g \equiv 0$ then means that these transformations will turn out to be isometries whenever they preserve the metric at one point. In our present situation this will be the case for the symmetries we are searching for to answer (P).

As $A_X$ is skew symmetric with respect to the Riemannian metric, we are left with the condition that $(\nabla_{VX}S)S\mathcal{H}$ be skew-symmetric as well in order to have $\bar{\nabla}_X g \equiv 0$. This immediately implies that it takes values in the horizontal distribution. We should have $g((\nabla_{VX}S)S\mathcal{H}Y, VZ) = -g(\mathcal{H}Y,(\nabla_{VX}S)S\mathcal{H}VZ) = 0$. Conversely, we claim that if $\mathcal{H}(\nabla_{VX}S)S\mathcal{H}Y) = (\nabla_{VX}S)S\mathcal{H}Y$, then $(\nabla_{VX}S)S\mathcal{H}Y$ is skew-symmetric.

Proof. $g(\mathcal{H}(\nabla_{VX}S)S\mathcal{H}Y, Z) = g(\mathcal{H}\nabla_{VX}S^2\mathcal{H}Y - \mathcal{H}S\nabla_{VX}S\mathcal{H}Y, Z)$

$= g(-\mathcal{H}\nabla_{VX}\mathcal{H}Y - \mathcal{H}S\nabla_{VX}S\mathcal{H}Y, Z) = g(\nabla_{VX}\mathcal{H}Y, -\mathcal{H}Z) - g(\nabla_{VX}S\mathcal{H}Y, \mathcal{H}S^{-1}Z)$

$= VXg(\mathcal{H}Y, -\mathcal{H}Z) - g(\mathcal{H}Y, -\nabla_{VX}\mathcal{H}Z) - VXg(S\mathcal{H}Y, \mathcal{H}S^{-1}Z) + g(S\mathcal{H}Y, \nabla_{VX}\mathcal{H}S^{-1}Z)$.

Recall that $S^{-1}|\mathcal{H} = -S$, and using the S-invariance of g we obtain

$= g(\mathcal{H}Y, \nabla_{VX}\mathcal{H}Z) - g(S\mathcal{H}Y, \nabla_{VX}\mathcal{H}SZ) = g(\mathcal{H}Y, \nabla_{VX}\mathcal{H}Z + \mathcal{H}S\nabla_{VX}\mathcal{H}SZ)$

$= -g(Y, \mathcal{H}\nabla_{VX}\mathcal{H}S^2Z - \mathcal{H}S\nabla_{VX}\mathcal{H}SZ) = -g(Y, \mathcal{H}(\nabla_{VX}S)S\mathcal{H}Z)$. ///

This shows that $\mathcal{H}(\nabla_{\mathcal{V}X}S)S\mathcal{H}$ is always skew-symmetric with respect to the Riemannian metric $g$ for any $X \in \chi(M)$. In particular, the condition $\overline{\nabla}g \equiv 0$ is satisfied if and only if $\mathcal{H}(\nabla_{\mathcal{V}X}S)S\mathcal{H}Y = (\nabla_{\mathcal{V}X}S)S\mathcal{H}Y$ for all $X$ and $Y$ in $\chi(M)$.

Since $S$ is nonsingular and preserves the distribution, we can express this condition as saying that $(\nabla_{\mathcal{V}X}S)\mathcal{H}$ is horizontally valued for all $X$. That is, $\mathcal{V}(\nabla_{\mathcal{V}X}S)\mathcal{H}Y = \mathcal{V}\nabla_{\mathcal{V}X}(S\mathcal{H}Y) - \mathcal{V}S\nabla_{\mathcal{V}X}(\mathcal{H}Y) = 0$, but $\mathcal{V}S = S\mathcal{V} = -I\mathcal{V} = -\mathcal{V}$, thus the above is equivalent to $\mathcal{V}(\nabla_{\mathcal{V}X}(S\mathcal{H}Y + \mathcal{H}Y)) = 0$ or $\mathcal{V}(\nabla_{\mathcal{V}X}[(I+S)\mathcal{H}Y]) = 0$. Again, $I + S$ is nonsingular on $\mathcal{H}$, thus it can be omitted, this yields $\mathcal{V}\nabla_{\mathcal{V}X}\mathcal{H}Y = 0$. Since $\mathcal{V}$ and $\mathcal{H}$ are orthogonal complementary distributions, this is the same as writing $\mathcal{V}\nabla_{\mathcal{V}X}(\mathcal{V}Y) = \nabla_{\mathcal{V}X}(\mathcal{V}Y)$ for all $X, Y \in X(M)$. (Here is the one row calculation of this fact: $g(\mathcal{H}\nabla_{\mathcal{V}X}\mathcal{V}Y, \mathcal{H}Z) = g(\nabla_{\mathcal{V}X}\mathcal{V}Y, \mathcal{H}Z) = -g(\mathcal{V}Y, \nabla_{\mathcal{V}X}\mathcal{H}Z))$, i.e., we have shown that the condition $\overline{\nabla}g \equiv 0$ is equivalent to the condition $\mathcal{V}\nabla_{\mathcal{V}X}(\mathcal{V}Y) = \nabla_{\mathcal{V}X}(\mathcal{V}Y)$.

This last condition has two interesting consequences, firstly, it is immediate that the vertical distribution is integrable: to see this we compute the Lie bracket

$[\mathcal{V}X, \mathcal{V}Y] = \nabla_{\mathcal{V}X}(\mathcal{V}Y) - \nabla_{\mathcal{V}Y}(\mathcal{V}X) = \mathcal{V}\nabla_{\mathcal{V}X}(\mathcal{V}Y) - \mathcal{V}\nabla_{\mathcal{V}Y}(\mathcal{V}X) = \mathcal{V}[\mathcal{V}X, \mathcal{V}Y]$.

i.e., the Lie bracket of two vertical fields is a vertical vector field. Secondly, if $F$ is a leaf of the foliation, then $F$ is a totally geodesic submanifold.

This is clear, since the condition $\mathcal{V}\nabla_{\mathcal{V}X}\mathcal{V}Y = \nabla_{\mathcal{V}X}\mathcal{V}Y$

precisely expresses that covariant differentiation of a vertical vector field with respect to another vertical vector field is a new vertical vector field.

Conversely, if $\mathcal{V}$ is an involutive distribution with leaves totally geodesic submanifolds, then $\mathcal{V}\nabla_{\mathcal{V}X}\mathcal{V}Y = \nabla_{\mathcal{V}X}\mathcal{V}Y$. (This is but the definition).

In conclusion we have

Proposition A . Let $(M,g)$ and $S$ be as in (P) above. Assume that $g$ is S-invariant, and let $\overline{\nabla}$ be the connection defined in (1). Then $g$ is parallel with respect to $\overline{\nabla}$, (i.e., $\overline{\nabla}g \equiv 0$) if and only if the vertical distribution defines a totally geodesic foliation of $M$.

As it was proved in Section 4, this foliation must have the following two properties. On one hand the leaves should be isometric to a Cartan symmetric space, and on the other hand, the foliation must be regular. We anticipate a little and note that since both connections $\nabla$ and $\overline{\nabla}$ coincide on the vertical distribution, it follows that $R = \overline{R}$ on $\mathcal{V}$ and also S-invariance will imply that $\overline{\nabla}\overline{R} = \nabla R = 0$ on $\mathcal{V}$. This says that the leaves are locally 2-symmetric spaces. The problem is then to see when they are globally symmetric. As for regularity, little is known to me.

For the rest of this section $V$ is assumed to be auto-parallel with respect to the Riemannian connection $\nabla$ (i.e., $\overline{\nabla}g \equiv 0$).

In the following lemma, we prove some relations that will be useful in our computations.

Lemma. Let U, V, W denote vertical vector fields, and H, K, L horizontal vector vields. Then

$$\text{(i)} \quad \overline{\nabla}_U V = \nabla_U V$$

$$\text{(ii)} \quad \overline{\nabla}_V H = \tfrac{1}{2}(\nabla_V H - S\nabla_V SH)$$

$$\text{(iii)} \quad \overline{\nabla}_H K = H\nabla_H K$$

$$\text{(iv)} \quad \overline{\nabla}_H V = V\nabla_H V.$$

Furthermore ., both $V$ and $H$ are auto-parallel with respect to $\overline{\nabla}$.

The proof is very simple, one only has to unwind the definitions. For example (i) is an immediate consequence of our assumption that $V$ be auto-parallel. It is also clear that $V$ and $H$ are auto-parallel with respect to $\overline{\nabla}$.

For the rest of the section, U, V, W will always denote vertical vector fields, and H, K. L horizontal ones.

We now study the condition $\overline{\nabla}S \equiv 0$.

The method we follow will be the same throughout the section. It consists of computing for each tensor under consideration each one of its components with respect to the vertical and horizontal distributions. Thus for $\overline{\nabla}S$ we have:

$(\overline{\nabla}_V S)W = \overline{\nabla}_V(SW) - S(\overline{\nabla}_V W) = -\overline{\nabla}_V W - (-\overline{\nabla}_V W) = 0$ where we have used the condition $\mathcal{V}$ auto-parallel with respect to $\overline{\nabla}$. $(\overline{\nabla}_V S)H = \overline{\nabla}_V(SH) - S(\overline{\nabla}_V H) = \frac{1}{2}(\nabla_V(SH) - S\nabla_V(SSH)) - \frac{1}{2}\, S(\nabla_V H - S\nabla_V(SH))$ (by (ii) of the above lemma) $= \frac{1}{2}(\nabla_V(SH) + S(\nabla_V H)) - \frac{1}{2}(S(\nabla_V H) + \nabla_V(SH)) = 0$. $(\overline{\nabla}_H S)V = 0$ as a consequence that $\mathcal{H}$ is auto-parallel with respect to $\overline{\nabla}$. Finally we have to compute $(\overline{\nabla}_H S)K = \overline{\nabla}_H(SK) - S(\overline{\nabla}_H K)$ using (iv) above, we have that this can be written as $\mathcal{H}\nabla_H(SK) - \mathcal{H}S(\nabla_H K)$ or simply as $\mathcal{H}(\nabla_H S)K$. That is $(\overline{\nabla}_H S)K = \mathcal{H}(\nabla_H S)K$. In conclusion we have:

Proposition B. Let $(M,g)$ and $S$ be as in (P), assume further that $\mathcal{V}$ is auto-parallel with respect to the Riemannian connection $\nabla$. Then $\overline{\nabla} S \equiv 0$ if and only if $\mathcal{H}(\nabla_H S)K = 0$ for all $H$ and $K$ horizontal vector fields.

The condition $\overline{\nabla} S \equiv 0$ can thus be interpreted for the horizontal distribution as the equivalent of the Kähler condition. In fact, we can draw the following corollary:

Corollary. Let $N$ be any connected submanifold of a Riemannian 4-symmetric space $(M,g,s)$ whose tangent bundle is $S$-invariant and is contained in the horizontal distribution. Assume further that $N$ is complete totally geodesic. Then $N$ is a Hermitian 2-symmetric space with $S$ as its complex structure.

Proof. The totally geodesic condition implies $(\overline{\nabla}_H S)K = \mathcal{H}(\nabla_H S)K = (\nabla_H S)K$ for all $H$ and $K$ tangent to $N$, and since for a 4-symmetric space $\overline{\nabla}S \equiv 0$ holds true, then $(\nabla_H S)K = 0$ for all $H$ and $K$ tangent to $N$, i.e. $S$ defines a complex structure on $N$ which is Kähler. Furthermore, the squares of the symmetries define involutions that leave $S$ invariant and hence $N$ is Hermitian 2-symmetric as claimed. Of course here we are using the well-known fact that a complete totally geodesic submanifold whose tangent space is left-invariant by an isometry then the submanifold itself is left-invariant by the isometry.

There is one point that deserves some attention. From our presentation it appears as if the conditions $\overline{\nabla}g \equiv 0$ and $\overline{\nabla}S \equiv 0$ were related in one direction, that is, that using $\overline{\nabla}g \equiv 0$, then we arrive at the result that $\overline{\nabla}S$ vanishes identically if and only if $\mathcal{H}(\nabla_H S)K = 0$. Actually we have that $\overline{\nabla}S \equiv 0$ is a much stronger condition that the vanishing of $\overline{\nabla}g$. Concretely we prove:

Proposition B'. Let $(M,g)$ and $S$ be as in (P). $\nabla$ the Riemannian connection and $\overline{\nabla}$ as given in (1). Then $\overline{\nabla}S \equiv 0$ if and only if $\mathcal{V}$ is auto-parallel with respect to $\nabla$ and $\mathcal{H}(\nabla_H S)K = 0$ for all $H$ and $K$ horizontal vector fields. In particular we have that $\overline{\nabla}S \equiv 0$ implies $\overline{\nabla}g \equiv 0$.

Proof. Let $U$ and $V$ be any two vertical vector fields, then $(\overline{\nabla}_V S)U = \overline{\nabla}_V SU - S\overline{\nabla}_V U = -\nabla_V U - S\nabla_V U$, hence this vanishes if and only if $S\nabla_V U = -\nabla_V U$ i.e. if and only if $\nabla_V U$ remains vertical, which is precisely the condition that $\mathcal{V}$ be auto-parallel with respect to $\nabla$.

Having proved this, the rest of the calculations performed to prove Proposition B hold true, hence the result follows.

We now study the torsion $\overline{T}$ of the connection $\overline{\nabla}$. As pointed out at the end of Section 3, the main difference between this connection and the Riemannian connection is that whereas $T$ is always identically zero, $\overline{T}$ in general does not vanish. In fact, we remarked that if $\overline{T}$ vanishes identically, then $\nabla S$ also vanishes identically and hence the underlying Riemannian manifold is a Cartan space. Thus $\overline{T}$ is an interesting tensor, it measures how far is a 4-symmetric space from being a 2-symmetric space. As usual, $U$ and $V$ are vertical and $H$ and $K$ horizontal. The assumption remains the same: $\mathcal{V}$ auto-parallel with respect to $\nabla$. Note that both connections coincide on $\mathcal{V}$, hence $\overline{T}(U,V) = T(U,V) = 0$. Thus we only have to compute $\overline{T}(V,H)$ and $\overline{T}(H,K)$. An easy calculation from the definitions shows that

$$\overline{T}(V,H) = \frac{1}{2}(\nabla_V S)SH + A_H V \quad \text{and}$$

$$\overline{T}(H,K) = -\mathcal{V}[H,K].$$

Observe that $(\nabla_V S)SH = \nabla_V(S^2H) - S\nabla_V(SH) = -\nabla_V H - S\nabla_V(SH)$ is always $S$-invariant, because if we substitute $S^{-1}V$ by $V$ and $S^{-1}H$ by $H$ and apply $S$ on the left we obtain: $-S\nabla_{S^{-1}V}(S^{-1}H) - S^2\nabla_{S^{-1}V}(SS^{-1}H) = -S\nabla_V(SH) - \nabla_V H$, which is our original expression. Here we have used $S^{-1}H = -SH$ and our assumption $V$ auto-parallel with respect to $\nabla$. Hence we have that $\overline{T}(V,H)$ is S-invariant if and only if $A_H V$ is S-invariant. On the other hand, $\overline{T}(H,K)$ gives us precisely the measure of how far is $H$ from being integrable which is in accordance with our previous remarks since in this case, if $H$ is integrable, then it defines a complementary foliation of $V$ with leaves complete totally geodesic submanifolds with a natural Hermitian structure on them, and hence, our space would be (locally) the product of two Riemannian 2-symmetric spaces. In conclusion we have

Proposition C. The torsion tensor of the connection $\overline{\nabla}$ is given as follows:

$$\begin{aligned} \overline{T}(U,V) &= 0 \quad (V \text{ is assumed auto-parallel}) \\ \overline{T}(V,H) &= \tfrac{1}{2}(\nabla_V S)SH + A_H V \\ \overline{T}(H,K) &= -V[H,K] = -A_H K + A_K H. \end{aligned}$$

Furthermore, $\frac{1}{2}(\nabla_V S)SH$ is always S-invariant. Hence $\overline{T}$ is S-invariant if and only if both $A_H V$ and $-V[H,K]$ are S-invariant. In particular, if $A$ is S-invariant then $\overline{T}$ is S-invariant. ///

It should be pointed out that although the condition $\overline{\nabla}S \equiv 0$ was interpreted as some sort of Kählerian condition, this condition and the condition that $\overline{T}$ be S-invariant at first do not seem to be very geometrical, and it looks as if given one tensorial condition we are only obtaining a new one.

However, it must be remembered that we pursue a final set of conditions where all of them are interwoven. We still have to study the conditions that $\overline{R}$ be $S$-invariant, and that $\overline{\nabla}R \equiv 0$ and $\overline{\nabla}\overline{T} \equiv 0$.

<u>We now study the curvature tensor $\overline{R}$ of the connection</u> $\overline{\nabla}$. As usual $U$, $V$ and $W$ denote vertical vector fields, and $H$, $K$ and $L$ horizontal vector fields.

<u>Proposition D</u>. Let $(M,g)$ and $S$ be as in (P), and let $\overline{\nabla}$ and $\nabla$ be respectively the connection defined in (1) and the Riemannian connection. Assume $\overline{\nabla}S \equiv 0$. Then the curvature tensor $\overline{R}$ of $\overline{\nabla}$ satisfies the following relations

(a) $\overline{R}(U,V)W = R(U,V)W$

(b) $\overline{R}(U,V)H = \frac{1}{2} R(U,V)H - \frac{1}{2} SR(U,V)SH - \frac{1}{4}(\nabla_U S)(\nabla_V S)H + \frac{1}{4}(\nabla_V S)(\nabla_U S)H$

(c) $\overline{R}(U,H)V = \mathcal{V}R(U,H)V$

(d) $\overline{R}(V,H)K = \frac{1}{2} \mathcal{H}R(V,H)K - \frac{1}{2} \mathcal{H}SR(V,H)SK$

(e) $\overline{R}(H,K)V = \mathcal{V}R(H,K)V - A_H A_K V + A_K A_H V$

(f) $\overline{R}(H,K)L = \mathcal{H}R(H,K)L - A_H A_K L + A_K A_H L - \frac{1}{2}(\nabla_{\mathcal{V}[H,K]} S)SL.$

Before proving this, we state a proposition which gives the conditions under which $\overline{R}$ is $S$-invariant. Then we give a joint proof of both propositions.

<u>Proposition D'</u>. Given the same notation and assumptions as in Proposition D, assume further that the tensor field $A$ is $S$-invariant. Then $\overline{R}$ is $S$-invariant if and only if the

following conditions are satisfied:

($\alpha$) $\mathcal{V}R(U,H)V = 0$

($\beta$) $\mathcal{H}R(V,H)K = S\mathcal{H}R(V,H)SK$

($\gamma$) $\mathcal{V}R(H,K)V$ and $\mathcal{H}R(H,K)L$ both are S-invariant.

<u>Proof</u> (Of Propositions D and D'). The idea is the same as we have been using. We study each of the components of $\bar{R}$ separately.

Recall that the condition $\nabla S \equiv 0$ is equivalent to $\mathcal{V}$ being auto-parallel with respect to $\nabla$ and to $\mathcal{H}(\nabla_H S)K = 0$ (both conditions taken together, see Proposition B'). We shall use these conditions repeatedly without any further mention. Also we shall use relations like $SV = -V$, $S^{-1}V = -V$ and $S^{-1}H = -SH$.

(a) This follows because $\mathcal{V}$ is auto-parallel with respect to $\nabla$ and hence $\bar{\nabla}_V W = \nabla_V W$ remains vertical and we have $\bar{R}(U,V)W = R(U,V)W$. Since S restricted to $\mathcal{V}$ is minus the identity, it is clear that $R(U,V)W$ is S-invariant.

(b) By unwinding the definitions, the following relation can be obtained: $\bar{R}(U,V)H = \frac{1}{2}R(U,V)H - \frac{1}{2}SR(U,V)SH$

$$+ \tfrac{1}{4}[S\nabla_U\nabla_V(SH) - \nabla_U\nabla_V H] - \tfrac{1}{4}[S\nabla_U(S\nabla_V H) + \nabla_U(S\nabla_V(SH))]$$

$$- \tfrac{1}{4}[S\nabla_V\nabla_U(SH) - \nabla_V\nabla_U H] + \tfrac{1}{4}[S\nabla_V(S\nabla_U H) + \nabla_V(S\nabla_U(SH))].$$

We shall work with the second row (by skew-symmetry with the third). Note that $S\nabla_U\nabla_V(SH) - \nabla_U(S\nabla_V(SH)) = -(\nabla_V S)(\nabla_V(SH))$, and also $-\nabla_U\nabla_V H - S\nabla_U(S\nabla_V H) = (\nabla_U S)(S\nabla_V H)$, adding up these two expressions we obtain $-(\nabla_U S)(\nabla_V S)H$. From this the result follows. Note that the way in which we have associated the terms in our very first relation shows that $\overline{R}(U,V)H$ is S-invariant. (Also, we can recall the proof of Proposition C together with the final relations). However, we can give a straightforward proof of this fact. Note that we have to show that $\overline{R}(U,V)H = S\overline{R}(S^{-1}U,S^{-1}V)S^{-1}H = -S\overline{R}(U,V)SH$. Thus S-invariance reduces in this case to prove that $\overline{R}(U,V)$ commutes with S. (For this one has to observe that $\overline{R}(U,V)$ preserves the horizontal distribution $\mathcal{H}$). But that $\overline{R}(U,V)$ commutes with S is an immediate consequence of the fact that $(\overline{\nabla}_V S)H = 0$ which is part of our assumptions (see also the proof of Proposition B). Hence we have that $\overline{R}(U,V)H$ is S-invariant.

(c) $$\overline{R}(U,H)V = \overline{\nabla}_U\overline{\nabla}_H V - \overline{\nabla}_H\overline{\nabla}_U V - \overline{\nabla}_{[U,H]}V = \overline{\nabla}_U(\mathcal{V}\nabla_H V) - \overline{\nabla}_H\nabla_U V - \overline{\nabla}_{[U,H]}V = \mathcal{V}\nabla_U\nabla_H V - \mathcal{V}\nabla_H\nabla_U V - \overline{\nabla}_{\mathcal{V}[U,H]}V - \overline{\nabla}_{\mathcal{H}[U,H]}V = \mathcal{V}(\nabla_U\nabla_H V - \nabla_H\nabla_U V) - \nabla_{\mathcal{V}[U,H]}V - \mathcal{V}\nabla_{\mathcal{H}[U,H]}V = \mathcal{V}(\nabla_U\nabla_H V - \nabla_H\nabla_U V - \nabla_{[U,H]}V) = \mathcal{V}R(U,H)V.$$

As for the S-invariance of this term, we shall see that it yields condition ($\alpha$). $\overline{R}(U,H)V$ is S-invariant if and only if $\overline{R}(U,H)V = S\overline{R}(S^{-1}U,S^{-1}H)S^{-1}V = -S\overline{R}(U,SH)V$ but $\overline{R}(U,SH)$ preserves $\mathcal{V}$, hence we must have $\overline{R}(U,H)V = \overline{R}(U,SH)V$ or $\overline{R}(U,(I-S)H)V = 0$. The term $(I-S)$ can be omitted since it is nonsingular when restricted to $\mathcal{H}$, thus the last relation can be interpreted as saying that $\overline{R}(U,H)V$ is S-invariant if and only if $\overline{R}(U,H)V$ vanishes identically. By what we have already proved this is precisely equivalent to the vanishing of $\mathcal{V}R(U,H)V$, and hence condition ($\alpha$).

(d) Some routine calculations yield the following relation:

$$\overline{R}(V,H)K = \frac{1}{2}\mathcal{H}R(V,H)K - \frac{1}{2}\mathcal{H}S\nabla_V(S\nabla_H K) + \frac{1}{2}\mathcal{H}\nabla_H(S\nabla_V(SK)) + \frac{1}{2}S\nabla_{\mathcal{V}[V,H]}(SK) - \frac{1}{2}\mathcal{H}\nabla_{\mathcal{H}[V,H]}K.$$

To simplify this expression we now use our assumption $\overline{\nabla}S \equiv 0$. Thus in particular we have $\mathcal{H}(\nabla_H S)K = 0$. With this in mind the above expression reduces to

$$\overline{R}(V,H)K = \frac{1}{2}\mathcal{H}R(V,H)K - \frac{1}{2}\mathcal{H}SR(V,H)SK.$$

For S-invariance we show that $\overline{R}(V,H)K$ must vanish identically: If $\overline{R}(V,H)K$ is S-invariant, then in particular it is $S^2$-invariant i.e.

$$\overline{R}(V,H)K = S^2\overline{R}(S^{-2}V,S^{-2}H)(S^{-2}K) = S^2\overline{R}(V,H)K = -\overline{R}(V,H)K.$$

Thus $\overline{R}(V,H)K$ is S-invariant if and only if it vanishes identically. Using what we have just proved, we must have $\mathcal{H}R(V,H)K = \mathcal{H}SR(V,H)SK$ which is condition ($\beta$).

(e) $$\begin{aligned}\overline{R}(H,K)V &= \overline{\nabla}_H\overline{\nabla}_K V - \overline{\nabla}_K\overline{\nabla}_H V - \overline{\nabla}_{[H,K]}V = \overline{\nabla}_H(\mathcal{V}\nabla_K V) - \overline{\nabla}_K(\mathcal{V}\nabla_H V)\\ &\quad - \overline{\nabla}_{[H,K]}V = \mathcal{V}\nabla_H(\mathcal{V}\nabla_K V) - \mathcal{V}\nabla_K(\mathcal{V}\nabla_H V) - \overline{\nabla}_{\mathcal{V}[H,K]}V - \overline{\nabla}_{\mathcal{H}[H,K]}V\\ &= \mathcal{V}\nabla_H(\nabla_K V - \mathcal{H}\nabla_K V) - \mathcal{V}\nabla_K(\nabla_H V - \mathcal{H}\nabla_H V) - \nabla_{\mathcal{V}[H,K]}V - \mathcal{V}\nabla_{\mathcal{H}[H,K]}V\\ &= \mathcal{V}R(H,K)V - A_H A_K V + A_K A_H V.\end{aligned}$$

As we have already assumed that $A$ is $S$-invariant, the last two terms are also $S$-invariant, hence $\overline{R}(H,K)V$ is $S$-invariant if and only if $\mathcal{V}R(H,K)V$ is $S$-invariant.

(f) $$\overline{R}(H,K)L = \overline{\nabla}_H\overline{\nabla}_K L - \overline{\nabla}_K\overline{\nabla}_H L - \overline{\nabla}_{[H,K]}L = \mathcal{H}\nabla_H\mathcal{H}\nabla_K L - \mathcal{H}\nabla_K\mathcal{H}\nabla_H L - \mathcal{H}\nabla_{\mathcal{H}[H,K]}L - \overline{\nabla}_{\mathcal{V}[H,K]}L = \mathcal{H}\nabla_H(\nabla_K L - \mathcal{V}\nabla_K L) - \mathcal{H}\nabla_K(\nabla_H L - \mathcal{V}\nabla_H L) - \mathcal{H}\nabla_{\mathcal{H}[H,K]}L - \frac{1}{2}\nabla_{\mathcal{V}[H,K]}L + \frac{1}{2}S\nabla_{\mathcal{V}[H,K]}SL$$
$$= \mathcal{H}R(H,K)L - A_HA_KL + A_KA_HL + \frac{1}{2}\nabla_{\mathcal{V}[H,K]}L + \frac{1}{2}S\nabla_{\mathcal{V}[H,K]}SL.$$

Note that the last terms can be written as $-\frac{1}{2}(\nabla_{\mathcal{V}[H,K]}S)SL$, and so $\overline{R}(H,K)L = \mathcal{H}R(H,K)L - A_HA_KL + A_KA_HL - \frac{1}{2}(\nabla_{\mathcal{V}[H,K]}S)SL$.

For the $S$-invariance of this term, we have already shown (see the comments before Proposition C) that $(\nabla_V S)SH$ is $S$-invariant, and since $\mathcal{V}[H,K] = A_HK - A_KH$ we have that if $A$ is $S$-invariant, $\overline{R}$ is $S$-invariant if and only if $\mathcal{H}R(H,K)L$. This together with the last part of the proof of (e) accounts for ($\gamma$). The proofs of both propositions are now complete. ///

Remarks. As it was pointed out at the beginning of this section, a set of necessary conditions to solve problem (P) is given by the fact that the curvature tensor $R$ of the Riemannian connection $\nabla$ and all its covariant derivatives $\nabla^{(n)}R$ must be $S$-invariant. Here we shall only study the $S$-invariance of $R$ and see how it compares with the $S$-invariance of $\overline{R}$ as already given in Proposition D'.

Proposition D". Let $(M,g)$ and $S$ be as in (P). Assume that the vertical distribution $V$ is totally geodesic with respect to the Riemannian connection. Then the Riemannian curvature tensor $R$ is $S$-invariant if and only if the following set of conditions is satisfied.

(a) $R(U,V)H = -SR(U,V)SH$

(b) $\mathcal{H}R(U,H)V = -S\mathcal{H}R(U,SH)V$

(c) $\mathcal{V}R(V,H)K = \mathcal{V}R(V,SH)SK$

(d) $\mathcal{V}R(H,K)V = \mathcal{V}R(SH,SK)V$

(e) $\mathcal{H}R(H,K)L = -S\mathcal{H}R(SH,SK)SL$

(f) $\mathcal{V}R(U,H)V = \mathcal{H}R(V,H)K = \mathcal{H}R(H,K)V = \mathcal{V}R(H,K)L = 0$. ///

From this we have

Corollary 1. The same notation and assumptions apply as in Proposition D'. Assume further that $R$ is $S$-invariant. Then $\bar{R}$ is $S$-invariant.

Proof. We would have to check that conditions $(\alpha)$, $(\beta)$ and $(\gamma)$ of Proposition D' are satisfied, but this is trivial. ///

The advantage of this corollary is that the $S$-invariance of $\bar{R}$ is now easily stated in terms of the $S$-invariance of $R$. The drawback is that it is not equivalent. $S$-invariance of $R$ is stronger. Note also that $\bar{R}$ simplifies

Corollary 2. The notation and assumptions are as in Proposition D. Assume also that $R$ is $S$-invariant, then $\bar{R}$ is given as follows:

(a) $\bar{R}(U,V)W = R(U,V)W$

(b) $\bar{R}(U,V)H = R(U,V)H - \frac{1}{4}(\nabla_U S)(\nabla_V S)H + \frac{1}{4}(\nabla_V S)(\nabla_U S)H$

(c) & (d) $\bar{R}(U,H)V = \bar{R}(V,H)K = 0$

(e) $\bar{R}(H,K)V = \mathcal{V}R(H,K)V - A_H A_K V + A_K A_H V$

(f) $\bar{R}(H,K)L = \mathcal{H}R(H,K)L - A_H A_K L + A_K A_H L - \frac{1}{2}(\nabla_{\mathcal{V}[H,K]}S)SL.$

The remaining two conditions we have to work on are $\bar{\nabla}\bar{T} \equiv 0$ and $\bar{\nabla}\bar{R} \equiv 0$ which tell now that $\bar{\nabla}$ is invariant under parallelism. At this point, instead of following our general procedure we look at the relations for $\bar{T}$ and $\bar{R}$ given in Propositions C and D and note that it is more convenient to break them up into parts which should vanish separately. More concretely we have that $\bar{T}$ can be given in terms of the two tensors $(\nabla_{\mathcal{V}(\cdot)}S)S\mathcal{H}(\cdot)$ and $A$, and that $\bar{R}$ is given in terms of $R$ and these two tensors. Now then, each one of these two tensors is also $S$-invariant for a Riemannian 4-symmetric space, therefore, both of them must be $\bar{\nabla}$-parallel, i.e. we must have: $\bar{\nabla}[(\nabla_{\mathcal{V}(\cdot)}S)S\mathcal{H}(\cdot)] \equiv 0$ and $\bar{\nabla}A \equiv 0$. With this in mind, we can state the following:

<u>Proposition E</u>. Let $(M,g)$ and $S$ be as in (P), and let $\bar{\nabla}$ be as given in (1). Assume that $\bar{\nabla}S \equiv 0$ and that both $A$ and $R$ are $S$-invariant. Then $\bar{\nabla}$ is invariant under parallelism (i.e. $\bar{\nabla}\bar{R} \equiv 0$ and $\bar{\nabla}\bar{T} \equiv 0$) if and only if the following conditions are satisfied:

(a) $\overline{\nabla} A \equiv 0$

(b) $\overline{\nabla}((\nabla_{V(\cdot)}S)SH(\cdot)) \equiv 0$

(c) $(\overline{\nabla}R)(U,V)W \equiv 0$

(d) $(\overline{\nabla}R)(U,V)H \equiv 0$

(e) $(\overline{\nabla}VR)(H,K)V \equiv 0$

(f) $(\overline{\nabla}HR)(H,K)L \equiv 0$

Proof. From the previous remarks and the formulas for $\overline{T}$ and $\overline{R}$ it is clear that if conditions (a) through (e) are satisfied, then both $\overline{R}$ and $\overline{T}$ are $\overline{\nabla}$-parallel.

Assume now that $\overline{\nabla}$ is invariant under parallelism. Then we claim that $(M,g)$ becomes a Riemannian 4-symmetric space with symmetry tensor $S$. To prove this we have to see that conditions (i) and (ii) in Section 3 are satisfied. (i) is immediate from Proposition C, Corollary 1 and our assumptions. To prove (ii), it remains to show that the condition $\overline{\nabla} g \equiv 0$ is satisfied. But this follows from $\overline{\nabla} S \equiv 0$ (see Proposition B').

In Section 3 we pointed out that any tensor field invariant under the group obtained by taking the closure, in the full group of isometries of $(M,g)$, of the group generated by the symmetries was $\overline{\nabla}$-parallel. Since the tensors we are considering satisfy this condition, the result follows. ///

Thus we have to study conditions (a),...,(e) of Proposition E. We start with (a).

Lemma 1. In addition to the assumption $\mathcal{V}$ auto-parallel with respect to $\nabla$, assume that A is S-invariant. Then $\overline{\nabla}A$ is given as follows:

$$\text{(a)}\quad (\overline{\nabla}_V A)_H K = \frac{1}{2}\, \mathcal{V}R(V,H)K - \frac{1}{2}\, \mathcal{V}R(V,SH)SK$$

$$\text{(b)}\quad (\overline{\nabla}_L A)_H K = \mathcal{V}(\nabla_L A)_H K$$

$$\text{(c)}\quad (\overline{\nabla}_V A)_H K = \frac{1}{2}\, \mathcal{H}R(V,H)W + \frac{1}{2}\, \mathcal{H}SR(V,SH)W$$

$$\text{(d)}\quad (\overline{\nabla}_K A)_H W = \mathcal{H}(\nabla_K A)_H W.$$

Proof. It should be pointed out that since both distributions $\mathcal{V}$ and $\mathcal{H}$ are orthogonal and auto-parallel with respect to $\overline{\nabla}$, and that $\overline{\nabla}$ is a metric connection, then it is only necessary to compute the above four components of $\overline{\nabla}A$, (the others vanish).

$$\text{(a)}\quad (\overline{\nabla}_V A)_H K = \overline{\nabla}_V(A_H K) - A_{\overline{\nabla}_V H}K - A_H\overline{\nabla}_V K$$
$$= \nabla_V A_H K - \frac{1}{2}\, A_{\nabla_V H}K + \frac{1}{2}\, A_{S\nabla_V SH}K - \frac{1}{2}\, A_H\nabla_V K + \frac{1}{2}\, A_H S\nabla_V SK.$$

In order to simplify this expression we first prove the following identities:

$$A_{S\nabla_V SH}K = \mathcal{V}\nabla_{\nabla_V SH}SK; \quad A_H S\nabla_V SK = \mathcal{V}\nabla_{SH}\nabla_V SK$$

and $\quad \mathcal{V}\nabla_V\nabla_{SH}SK = -\mathcal{V}\nabla_V\nabla_H K.$

We shall only prove the last one:

$$\mathcal{V}\nabla_V\nabla_{SH}SK = \nabla_V\mathcal{V}\nabla_{SH}SK = \nabla_V A_{SH}SK = \nabla_V SA_H K = -\nabla_V\mathcal{V}\nabla_H K = -\mathcal{V}\nabla_V\nabla_H K.$$

(The first two can be proved in an analogous way, it is only necessary to use the S-invariance of A).

It follows that $(\overline{\nabla}_V A)_H K$ can be rewritten as

$$\mathcal{V}\nabla_V\nabla_H K - \frac{1}{2}\,\mathcal{V}\nabla_{\nabla_V H}K + \frac{1}{2}\,\mathcal{V}\nabla_{\nabla_V SH}SK - \frac{1}{2}\,\mathcal{V}\nabla_H\nabla_V K + \frac{1}{2}\,\mathcal{V}\nabla_{SH}\nabla_V SK =$$

$$\frac{1}{2}\mathcal{V}R(V,H)K + \frac{1}{2}\,\mathcal{V}\nabla_{[V,H]}K + \frac{1}{2}\,\mathcal{V}\nabla_V\nabla_H K - \frac{1}{2}\,\mathcal{V}\nabla_{\nabla_V H}K + \frac{1}{2}\,\mathcal{V}\nabla_{\nabla_V SH}SK + \frac{1}{2}\,\mathcal{V}\nabla_{SH}\nabla_V SK.$$

Using the fact that $\nabla$ is torsion free the second and fourth term of the above expression combine to give $-\frac{1}{2}\,\mathcal{V}\nabla_{\nabla_V H}K$, the third identity proved above then allows us to rewrite the whole expression as follows

$$= \frac{1}{2}\,\mathcal{V}R(V,H)K - \frac{1}{2}\,\mathcal{V}\nabla_{\nabla_H V}K + \frac{1}{2}\,\mathcal{V}\nabla_{\nabla_V SH}SK + \frac{1}{2}\,\mathcal{V}R(SH,V)SK + \frac{1}{2}\,\mathcal{V}\nabla_{[SH,V]}SK$$

$$= \frac{1}{2}\,\mathcal{V}R(V,H)K - \frac{1}{2}\,\mathcal{V}\nabla_{\nabla_H V}K + \frac{1}{2}\,\mathcal{V}\nabla_{\nabla_{SH}V}SK + \frac{1}{2}\,\mathcal{V}R(SH,V)SK.$$

The term $\mathcal{V}\nabla_{\nabla_{SH}V}SK$ can be rewritten as follows: $\mathcal{V}\nabla_{\mathcal{V}\nabla_{SH}V}SK + \mathcal{V}\nabla_{\mathcal{H}\nabla_{SH}V}SK$ because of the totally geodesic condition of $\mathcal{V}$, the first term vanishes identically, whereas the second which is equal to $A_{A_{SH}V}SK$ can be written (because of the S-invariance of A) as $A_{A_H V}K$ or $\mathcal{V}\nabla_{\nabla_H V}K$ where for the last relation one uses once again the totally geodesic condition. Substitution of this expression in our last equality yields the desired relation.

(b) and (d) are straightforward. One only has to unwind the definitions.

Finally we prove (c): $(\overline{\nabla}_V A)_H W = \overline{\nabla}_V A_H W - A_{\overline{\nabla}_V H} W - A_H \overline{\nabla}_V W =$

$$= \frac{1}{2}(\nabla_V A_H W - S\nabla_V (SA_H W)) - \frac{1}{2}(A_{\nabla_V H} W - A_{S\nabla_V SH} W) - A_H \nabla_V W.$$

Using the S-invariance of $A$ and the totally geodesic condition on $V$ this can be written as

$$= \frac{1}{2}(\nabla_V (H\nabla_H W) - S\nabla_V (H\nabla_{SH} SW)) - \frac{1}{2}(H\nabla_{\nabla_V H} W + SH\nabla_{\nabla_V SH} W) - H\nabla_H \nabla_V W.$$

The projection $H$ can be taken out of all the parentheses and associating we have

$$= \frac{1}{2}\, HR(V,H)W + \frac{1}{2}\, H\nabla_{[V,H]} W - \frac{1}{2}\, H\nabla_H \nabla_V W + \frac{1}{2}\, HS\nabla_V \nabla_{SH} W$$
$$- \frac{1}{2}\, H\nabla_{\nabla_V H} W - \frac{1}{2}\, SH\nabla_{\nabla_{SH} V} W - \frac{1}{2}\, SH\nabla_{[V,SH]} W.$$

The last two terms correspond to the second term in the second parenthesis by means of the vanishing of the torsion of $\nabla$. Note the following two identities:

$$S^2 H\nabla_H \nabla_V W = SH\nabla_{SH}(S\nabla_V W) = -SH\nabla_{SH} \nabla_V W \quad \text{and}$$

$$H\nabla_{\nabla_H V} W = H\nabla_{H\nabla_H V} W = A_{A_H V} W = -SA_{A_{SH} V} W = -HS\nabla_{\nabla_{SH} V} W.$$

The third equality follows by S-invariance of $A$. Hence, the third, fourth and seventh terms of our above relation combine to give $\frac{1}{2}\, SHR(V,SH)W$ whereas the second, fifth and sixth add up to zero. This proves (d). ///

As an immediate consequence we have:

Corollary. The same hypothesis as in the lemma. Assume also that R is S-invariant. Then A is $\bar{\nabla}$-parallel if and only if

(a) $\mathcal{V}(\nabla_L A)_H K = 0$

(b) $\mathcal{H}(\nabla_K A)_H W = 0.$ ///

We now study condition (b) of Proposition E.

Lemma 2. Assume $\bar{\nabla}S \equiv 0$. Then
$[\bar{\nabla}_H(\nabla S)]_V K = \mathcal{H}R(H,V)SK - \mathcal{H}SR(H,V)K.$

Proof. $[\bar{\nabla}_H(\nabla S)]_V K = \bar{\nabla}_H(\nabla_V SK - S\nabla_V K) - (\nabla_{\bar{\nabla}_H V} SK - S\nabla_{\bar{\nabla}_H V} K) -$

$(\nabla_V(S\bar{\nabla}_H K) - S\nabla_V\bar{\nabla}_H K).$

This expression can be rewritten as follows. (Here is one point where one has to use the assumption $\bar{\nabla}S \equiv 0$)

$$= \mathcal{H}R(H,V)SK + \mathcal{H}\nabla_{[H,V]}SK + \mathcal{H}SR(V,H)K + \mathcal{H}S\nabla_{[V,H]}K$$
$$- \nabla_{\mathcal{V}\nabla_H V}SK + S\nabla_{\mathcal{V}\nabla_H V}K.$$

Since $[H,V] - \mathcal{V}\nabla_H V = \mathcal{H}[H,V] + \mathcal{V}\nabla_V H = \mathcal{H}[H,V]$ (recall that $\bar{\nabla}S \equiv 0$ implies that $\mathcal{V}$ is totally geodesic), the above relation can be written as

$$= \mathcal{H}R(H,V)SK + \mathcal{H}SR(V,H)K + \mathcal{H}\nabla_{\mathcal{H}[H,V]}SK - \mathcal{H}S\nabla_{\mathcal{H}[H,V]}K.$$

Finally, once again use the fact that $\bar{\nabla}S \equiv 0$ implies $\nabla_H SK = S\nabla_H K$ to obtain the desired formula. ///

Remarks. If $R$ is $S$-invariant, then the term we have just computed vanishes identically. In this case, the remaining term to be computed is given by $[\overline{\nabla}_W(\nabla S)]_V K$ however I have not succeeded in finding a curvature relation like the one we obtained for the other component.

In the following two lemmas we study conditions (c) and (d) of Proposition E.

Lemma 3. Assume $\mathcal{V}$ auto-parallel with respect to $\nabla$ and $R$ $S$-invariant. Then

$$(\alpha) \quad (\overline{\nabla}_{V_0} R)(U,V)W = (\nabla_{V_0} R)(U,V)W$$

$$(\beta) \quad (\overline{\nabla}_H R)(U,V)W = \mathcal{V}(\nabla_H R)(U,V)W.$$

Proof. $(\alpha)$ is immediate from the totally geodesic condition for $\mathcal{V}$. As for $(\beta)$ one only has to recall that $\overline{\nabla}_H V = \mathcal{V}\nabla_H V$ and use the following relations: $R(\mathcal{V}X,U)V = \mathcal{V}R(X,U)V$, $R(U,V)\mathcal{V}X = \mathcal{V}R(U,V)X$. We prove the first:

$R(\mathcal{V}X,U)V = R(\frac{1}{2}(I+S^2)X,U)V = \frac{1}{2}R(X,U)V + \frac{1}{2}R(S^2X,U)V$ by $S$-invariance $R(S^2X,U)V = R(X,U)V$ and the result follows. ///

As a consequence we have that $\mathcal{V}$ has to be a foliation by locally 2-symmetric spaces.

Lemma 4. The same hypothesis as above

$$(\alpha) \quad (\overline{\nabla}_W R)(U,V)H = (\nabla_W R)(U,V)H + \frac{1}{2}(\nabla_W S)SR(U,V)H - \frac{1}{2}R(U,V)(\nabla_W S)SH$$

($\beta$) $(\overline{\nabla}_K R)(U,V)H = \mathcal{H}(\nabla_K R)(U,V)H.$

We omit the proof and only point out that to prove ($\beta$) the following relation is used $R(\mathcal{V}\nabla_K U,V)H = \mathcal{H}R(\nabla_K U,V)H.$ ///

In the following lemma we study condition (e) of Proposition E.

Lemma 5. Assume $\mathcal{V}$ auto-parallel with respect to $\nabla$ and $R$ S-invariant. Then

($\alpha$) $[\overline{\nabla}_W(\mathcal{V}R)](H,K)V = \frac{1}{2}[\nabla_W(\mathcal{V}R)](H,K)V + \frac{1}{2}[\nabla_W(\mathcal{V}R)](SH,SK)V$

($\beta$) $[\overline{\nabla}_L(\mathcal{V}R)](H,K)V = \mathcal{V}[\nabla_L(\mathcal{V}R)](H,K)V.$

Proof. ($\alpha$) $[\overline{\nabla}_W(\mathcal{V}R)](H,K)V = \overline{\nabla}_W(\mathcal{V}R(H,K)V) - \mathcal{V}R(\overline{\nabla}_W H,K)V -$

$\mathcal{V}R(H,\overline{\nabla}_W K)V - \mathcal{V}R(H,K)\overline{\nabla}_W V = \nabla_W(\mathcal{V}R(H,K)V) - \frac{1}{2}\mathcal{V}R(\nabla_W H,K)V +$

$\frac{1}{2}\mathcal{V}R(S\nabla_W SH,K)V - \frac{1}{2}\mathcal{V}R(H,\nabla_W K)V + \frac{1}{2}\mathcal{V}R(H,S\nabla_W SK)V - \mathcal{V}R(H,K)\nabla_W V.$

By the S-invariance of $R$ we have $\mathcal{V}R(H,SX)V = -\mathcal{V}R(SH,X)V$ and $\mathcal{V}R(SH,SK)V = \mathcal{V}R(H,K)V$, it is now easy to see how the result follows.

($\beta$) $[\overline{\nabla}_L(\mathcal{V}R)](H,K)V = \mathcal{V}\nabla_L(\mathcal{V}R(H,K)V) - \mathcal{V}R(\mathcal{H}\nabla_L H,K)V -$

$\mathcal{V}R(H,\mathcal{H}\nabla_L K)V - \mathcal{V}R(H,K)\mathcal{V}\nabla_L V.$

We work with the term $\mathcal{V}R(\mathcal{H}\nabla_L H,K)V$, recall that $\mathcal{H} = \frac{1}{2}(I-S^2)$, a substitution and using S-invariance of $R$ we obtain that the term is equal to $\mathcal{V}R(\nabla_L H,K)V$. It is also easy to see that $R(H,K)\circ\mathcal{V} = \mathcal{V}\circ R(H,K)$ (use S-invariance of $R$). Hence our above relation becomes

$= \mathcal{V}\nabla_L(\mathcal{V}R(H,K)V) - \mathcal{V}R(\nabla_L H,K)V - \mathcal{V}R(H,\nabla_L K)V - \mathcal{V}R(H,K)\nabla_L V.$

From this, the desired result follows. ///

The next lemma gives a description of condition (f) of Proposition E.

Lemma 6. Assume that $\mathcal{V}$ is auto-parallel with respect to $\nabla$ and that $R$ is S-invariant. Then the following identities hold true.

($\alpha$) $[\overline{\nabla}_V(\mathcal{H}R)](H,K)L = \frac{1}{2}\,\mathcal{H}(\nabla_V R)(H,K)L - \frac{1}{2}\,\mathcal{H}S(\nabla_V R)(SH,SK)SL$

($\beta$) $[\overline{\nabla}_{H_0}(\mathcal{H}R)](H,K)L = \mathcal{H}[\nabla_{H_0}(\mathcal{H}R)](H,K)L.$

Proof. ($\alpha$) is almost immediate, one only has to use the S-invariance of $R$ in a straightforward fashion.

($\beta$) For this use the relations $\mathcal{H}R(\mathcal{H}X,H)K = \mathcal{H}R(X,H)K$ and $\mathcal{H}R(H,K)\mathcal{H} = \mathcal{H}R(H,K)$ which follow immediately from the S-invariance of $R$ (cf. the proof of our previous lemma). ///

We can now restate Proposition E as follows:

Proposition E'. Let $(M,g)$ and $S$ be as in (P), and let $\overline{\nabla}$ be as given in (1). Assume that $\overline{\nabla}S \equiv 0$ and that both $A$ and $R$ are S-invariant. Then $\overline{\nabla}$ is invariant under parallelism if and only if the following conditions are satisfied.

(a) 1. $\mathcal{V}(\nabla_L A)_H K = 0$ 2. $\mathcal{H}(\nabla_K A)_H W = 0$

(b) $[\overline{\nabla}_W(\nabla S)]_V K = 0$

(c) 1. $V(\nabla_{V_0} R)(U,V)W = 0$ (i.e. The leaves of $V$ are Riemannian locally 2-symmetric spaces)

2. $V(\nabla_H R)(U,V)W = 0$

(d) 1. $(\nabla_W R)(U,V)H + \frac{1}{2}(\nabla_W S)SR(U,V)H - \frac{1}{2} R(U,V)(\nabla_W S)SH = 0$

2. $H(\nabla_K R)(U,V)H = 0$

(e) 1. $[\nabla_W(VR)](H,K)V + [\nabla_W(VR)](SH,SK)V = 0$

2. $V[\nabla_L(VR)](H,K)V = 0$

(f) 1. $H(\nabla_V R)(H,K)L - HS(\nabla_V R)(SH,SK)SL = 0$

2. $H[\nabla_{H_0}(HR)](H,K)L = 0$ ///

Comments. The fact that for a Riemannian 4-symmetric space R and all its covariant derivatives $\nabla^{(n)}R$ must be preserved by S can be used to restate Proposition E'. We shall do this next. For this we shall only use the S-invariance of $\nabla R$. At this point it is clear which of the components of $\nabla R$ vanish, so we leave it to the reader to write explicitly the proposition for $\nabla R$ analogous to Proposition D".

Proposition E". The same notation and conditions as in Proposition E'. Assume further that $\nabla R$ is S-invariant. Then $\overline{\nabla}$ is invariant under parallelism if and only if the following conditions are satisfied.

($\alpha$) 1. $V(\nabla_L A)_H K = 0$ 2. $H(\nabla_K A)_H W = 0$

($\beta$) $[\overline{\nabla}_W(\nabla S)]_V K = 0$

$$(\gamma) \quad (\nabla_W R)(U,V)H + \frac{1}{2}(\nabla_W S)SR(U,V)H - \frac{1}{2}\, R(U,V)(\nabla_W S)SH = 0.$$

Proof. We have to check that conditions (a) through (f) of Proposition E' are satisfied whenever ($\alpha$), ($\beta$) and ($\gamma$) are satisfied. ($\alpha$) and ($\beta$) account for (a) and (b), ($\gamma$) accounts for (d.1). Thus we have to show that the remaining set of conditions follows from the S-invariance of $\nabla R$. That this is so is clear for conditions (c), (d.2) and (f.1). We now have to work on conditins (e) and (f.2).

We start with (e). From S-invariance we have

$$(\nabla_W R)(H,K)V = S^{-1}(\nabla_{SW} R)(SH,SK)SV = S^{-1}(\nabla_W R)(SH,SK)V$$

and since we are taking covariant derivative with respect to a vertical vector field, if we take the vertical and horizontal components they will be preserved, in fact we have $\nabla_V(\mathcal{V}R) = \mathcal{V}\nabla_V R$ and $\nabla_V(\mathcal{H}R) = \mathcal{H}\nabla_V R$, hence each one of these components must be S-invariant separately. Apply this to the vertical component to obtain the result, i.e. from the above relation we have

$$[\nabla_W(\mathcal{V}R)](H,K)V = S^{-1}[\nabla_W(\mathcal{V}R)](SH,SK)V = -[\nabla_W(\mathcal{V}R)](SH,SK)V.$$

Therefore (e.1) holds true.

As for (e.2), we know that $\mathcal{V}(\nabla_L R)(H,K)V = 0$ (since it has three horizontal components), it is also easy to see that it decomposes as follows:

$$\mathcal{V}(\nabla_L R)(H,K)V = \mathcal{V}[\nabla_L(\mathcal{V}R)](H,K)V + \mathcal{V}\nabla_L(\mathcal{H}R(H,K)V)$$

we claim that each of these two terms is S-invariant and hence they vanish. It is only necessary to prove the S-invariance of one of them:

$$S^{-1}\mathcal{V}\nabla_{SL}(\mathcal{H}R(SH,SK)SV) = S^{-1}\mathcal{V}\nabla_{SL}(S\mathcal{H}R(H,K)V) \text{ (by S-invariance of R).}$$

Note that this can be written as $S^{-1}A_{SL}(S\mathcal{H}R(H,K)V)$ and since we have assumed that A is S-invariant, the result follows.

The proof of (f.2) is similar to this one, here we have that $\mathcal{H}(\nabla_{H_0} R)(H,K)L = 0$ (odd number of horizontal components), decompose this tensor as $\mathcal{H}[\nabla_{H_0}(\mathcal{H}R)](H,K)L + \mathcal{H}\nabla_{H_0}(\mathcal{V}R(H,K)L)$. Note that the second term is equal to $A_{H_0}(\mathcal{V}R(H,K)L)$ from which it follows that it is S-invariant and hence that must vanish. ///

We summarize all the above results in the following theorems.

<u>Theorem</u>. Let $(M,g)$ be a complete, connected, simply connected Riemannian manifold. Let S be a tensor field on M of type (1.1), of order four, with no eigenvalue $\pm 1$ that preserves g.

(i) Define $\mathcal{V}$ as the distribution for the eigenvalue +1 of $S^2$ and $\mathcal{H}$ as the orthogonal complementary distribution. Then $\mathcal{V}$ defines a totally geodesic foliation if and only if the connection

$$\bar{\nabla}_X Y = \nabla_X Y - A_X Y + \frac{1}{2}(\nabla_{\mathcal{V}X} S)S\mathcal{H}Y$$

is a metric connection, i.e. $\bar{\nabla} g \equiv 0$. (Here $A_X Y = \mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{H}X}\mathcal{H}Y$ and $\mathcal{V}$ and $\mathcal{H}$ denote the orthogonal projections onto the distributions).

(ii) The tensor field $S$ is parallel with respect to $\overline{\nabla}$ if and only if $V$ defines a totally geodesic foliation with respect to $\nabla$ and $H(\nabla_H S)K = 0$ for all $H$ and $K$ horizontal vector fields (i.e. lying in $H$) ("Kähler condition" for $H$).

(iii) Assume $\overline{\nabla}S \equiv 0$ and that $A$, $R$ and $\nabla R$ are $S$-invariant. Then the connection $\overline{\nabla}$ is invariant under parallelism (i.e. $\overline{\nabla}\overline{R} \equiv 0$ and $\overline{\nabla}\overline{T} \equiv 0$) if and only if the following three conditions are satisfied:

($\alpha$) 1. $V(\nabla_L A)_H K = 0$ 2. $H(\nabla_K A)_H W = 0$

($\beta$) $[\overline{\nabla}_W(\nabla S)]_V K = 0$

($\gamma$) $(\nabla_W R)(U,V)H + \frac{1}{2}(\nabla_W S)SR(U,V)H - \frac{1}{2} R(U,V)(\nabla_W S)SH = 0.$ ///

It is now immediate the local characterization of Riemannian 4-symmetric spaces.

<u>Theorem</u>. The same notation as above. Then $(M,g)$ is a Riemannian 4-symmetric space with symmetry tensor $S$ if and only if the following conditions are satisfied.

1. The distribution $V$ defines a totally geodesic foliation with respect to the Reimannian connection.

2. $H(\nabla_H S)K = 0$ for all $H$ and $K$ vector fields on $H$ (Kähler condition).

3. $A$, $R$ and $\nabla R$ are $S$-invariant

4. Conditions ($\alpha$), ($\beta$) and ($\gamma$) of (iii) of the above theorem are satisfied. ///

# CHAPTER II

## CLASSIFICATION OF COMPACT RIEMANNIAN 4-SYMMETRIC SPACES

### §7. De Rham Decomposition

In Riemannian geometry, as in mathematics in general, one of the fundamental problems is that of classification. That is, given a class of spaces, then write down a list of these spaces that completely exhausts all the possibilities. Our task in this and the next sections is to find a method that allows us to classify Riemannian 4-symmetric spaces. Here we shall only consider the compact simply connected case since it is technically less difficult than the general case. The idea we follow is essentially the same as that used by Cartan to classify Riemannian 2-symmetric spaces. The geometric problem is translated into an equivalent problem in terms of Lie algebras, and then by means of the theory of root systems worked out.

The basic building blocks in Riemannian geometry are the irreducible Riemannian manifolds, i.e. manifolds whose holonomy group at a point $p$ - and hence at any other point $q$ - acts irreducibly on the tangent space. Thus, towards a classification of the Riemannian 4-symmetric spaces, it is first necessary to describe their irreducible components. In this respect, the most general result has already been proved by Kowalski [13]. We quote it:

*Theorem*. Let $(M,g)$ be a simply connected Riemannian n-symmetric space and let $M = M_0 \times M_1 \times \ldots \times M_r$ be its de Rham decomposition, where $M_0$ is a Euclidean space and $M_1,\ldots,M_r$ are irreducible. Then each $M_i$ is a Riemannian $n_i$-symmetric space with $n_i$ a divisor of $n$, $i = 1,\ldots,r$.

The idea of the proof is very simple. First note that since $M$ is homogeneous, it is complete and the de Rham decomposition theorem can be applied. Let $p$ be a point in $M$, and let $T_pM = V_0 \oplus V_1 \oplus \ldots \oplus V_r$ be the orthogonal decomposition of the tangent space into irreducible components (with respect to the holonomy group action), then for each $i$, $1 \leq i \leq r$, there exists an integer $k_i$ such that $(s_p)_*^{k_i}(V_i) = V_i$ and $(s_p)_*^{k_i}$ restricted to $V_i$ has no eigenvalue $+1$. Since each $M_i$ is complete, it follows that $(s_p)^{k_i}$ preserves $M_i$, defining in this form a symmetry at $p$ of order $n_i$ with $n_i k_i = n$. Then using the identity component of the closure of the group generated by the symmetries $(s_p)$ it is possible to obtain symmetries for $M_i$ at any other point which satisfy the regularity condition.

For the particular case $n = 4$, the theorem yields the de Rham decomposition of a Riemannian 4-symmetric space.

*Theorem*. Let $(M,g)$ be a simply connected Riemannian 4-symmetric space, and let $M = M_0 \times M_1 \times \ldots \times M_r$ be its de Rham decomposition where $M_0$ is a Euclidean space and $M_1,\ldots,M_r$ are irreducible. Then each of the components is either a Riemannian 2-symmetric space or a Riemannian 4-symmetric space.

The outline of the proof of the theorem shows that some powers of the symmetries preserve the components $M_i$ of $M$. We now show that for a Riemannian 4-symmetric space more can be said, that is we prove that the symmetries $(s_p)$ preserve the de Rham decomposition.

Proof. With the above notation we have to prove that $k_i = 2$ can never happen. If there exists an $i$ for which $k_i = 2$, then $(s_p)^2$ restricted to $M_i$ is an isometry of $M_i$ of order two with $p$ as an isolated fixed point and $M_i$ becomes a Cartan symmetric space. Then $M_i \times s_p(M_i)$ is invariant under $s_p$ with both factors Cartan symmetric spaces. Then, the symmetry tensor $S$ defines an almost complex structure when restricted to the tangent bundle $T(M_i \times s_p(M_i))$. But since $S$ is invariant under the symmetries, it is invariant under the involutions of each of the factors. Hence $M_i \times s_p(M_i)$ is a Hermitian 2-symmetric space. Therefore its de Rham components are also Hermitian 2-symmetric spaces. Furthermore, their complex structures are induced by the restriction of the complex structure on $M_i \times s_p(M_i)$ to each of the factors. In other words, the symmetry tensor $S$ preserves each of the tangent bundles $TM_i$ and $T(s_pM_i)$. This gives a contradiction since it was assumed $k_i = 2$. ///

The significance of the above result lies in the fact that if now $G$ is the identity component of the closure of the subgroup generated by the symmetries, then $G$ preserves the de Rham decomposition and hence decomposes as a product $G_0 \times G_1 \times \ldots \times G_r$ with each $G_i$ acting transitively on $M_i$. Hence we have $M = G/K$ decomposes as $M = M_0 \times M_1 \times \ldots \times M_r$ and also $M = G/K = G_0/K_0 \times G_1/K_1 \times \ldots \times G_r/K_r$ with $M_i = G_i/K_i$, $0 \le i \le r$ and $G_i$ plays the role of $G$ for the factor $M_i$.

Thus the classification of Riemannian 4-symmetric spaces is equivalent to the classification of homogeneous spaces of the form $G/K$ with $G$ endowed with an automorphism $\sigma$ of order four such that $G_0^\sigma \subset K \subset G^\sigma$ and $G/K$ Riemannian homogeneous with a $\sigma$-invariant inner product (see Proposition in §2). However we should have the requirement that $G/K$ be irreducible as a Riemannian manifold.

Here the situation differs from that for Riemannian 2-symmetric spaces. For these spaces the isotropy group $K$ acting on $T_0M$ coincides with the action of the holonomy group. Thus an explicit criterion can be given between irreducibility of the space $(M,g)$ as a Riemannian manifold, and of $G/K$ as a homogeneous space.

As for Riemannian 4-symmetric spaces, there does not seem to be any relation, at first, between irreducibility in the above two senses. It looks as if the problem of classification should start by being a topological one, i.e., classify all the possible irreducible homogeneous spaces $G/K$, with $G$ and $K$ as above, do this, without any reference to a particular Riemannian metric, and then give some sort of information with respect to the metrics that such a homogeneous space can bear so as to become a Riemannian 4-symmetric space.

The classification of the simply connected homogeneous spaces $G/K$ with $G_0^\sigma \subset K \subset G^\sigma$, $\sigma$ an automorphism of order four on $G$, is equivalent to the classification of the pairs $(\mathfrak{g}, \sigma)$ with $\mathfrak{g}$ a Lie algebra over $\mathbb{R}$ and $\sigma$ an automorphism of order four. Thus our objective will be to classify the pairs $(\mathfrak{g}, \sigma)$. The problem in general is a technically difficult one, hence we restrict ourselves to the case when $G/K$ is a compact manifold. Since we also want it to be Riemannian homogeneous, it is natural to ask that $K$ be compact. All this in turn is equivalent to asking that the group $G$ be compact and semisimple. And using Weyl's theorem is equivalent to assuming that the Lie algebra $\mathfrak{g}$ be compact and semisimple. Thus our aim is the classification of the automorphisms of order four of compact semisimple Lie algebras over $\mathbb{R}$.

## §8. Automorphisms of Order Four of Compact Semisimple Lie Algebras

We set about classifying the pairs $(\mathfrak{g}, \sigma)$ with $\mathfrak{g}$ a compact semisimple Lie algebra and $\sigma$ an automorphism of order four. We accomplish this in two steps, first we consider the case when $\sigma$ is an inner automorphism, and then the case when $\sigma$ is an outer automorphism. For the first part we follow the work by Wolf and Gray [28]. Whereas for the second part we mainly follow the general theory as in Helgason [9].

We feel that even though the method in Helgason is rather more general - it yields a classification for both inner and outer automorphisms of finite order at once, the method by Wolf and Gray is rather straightforward and gives the same information without having to resort to more general and abstract constructions, in fact, their method has the advantage of giving in a very explicit fashion - as we shall see - the form of the inner automorphism in terms of the elements of a maximal abelian subalgebra of $\mathfrak{g}$.

A complete and detailed exposition of these methods would require a treatise on the subject. Since most of the general theory is already written, we start by briefly summarizing the main result that will be needed for the classification. Along the way we explain the notation that will be used. Finally, we solve the technical points for the case of symmetries of order four, and draw the corresponding tables. The basic references are the two sources already mentioned above. We mainly adopt Helgason's terminology and notation.

<u>An equivalence relation for the set of pairs $(\mathfrak{g},\sigma)$.</u> Given a pair $(\mathfrak{g},\sigma)$ with $\mathfrak{g}$ a compact semisimple Lie algebra and $\sigma$ an automorphism of order four, the corresponding compact simply connected homogeneous space is constructed as the quotient space $\tilde{G}/\tilde{K}$ where $\tilde{G}$ is the (compact) simply connected Lie group with Lie algebra $\mathfrak{g}$, and $\tilde{K}$ is the necessarily connected fixed point set of the induced automorphism $\tilde{\sigma}$ on $\tilde{G}$. $\tilde{G}/\tilde{K}$ can be endowed with certain metric - not necessarily unique - so as to become a Riemannian 4-symmetric space. The natural question is now: when do two given pairs $(\mathfrak{g}_1,\sigma_1)$ and $(\mathfrak{g}_2,\sigma_2)$ give rise to equivalent manifolds? If there exists an isomorphism $\phi$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that

$$\begin{tikzcd} \mathfrak{g}_1 \arrow[r, "\phi"] \arrow[d, "\sigma_1"'] & \mathfrak{g}_2 \arrow[d, "\sigma_2"] \\ \mathfrak{g}_1 \arrow[r, "\phi"] & \mathfrak{g}_2 \end{tikzcd}$$

is a commutative diagram, then $\phi$ induces an isomorphism $\tilde{\phi}:\tilde{G}_1 \to \tilde{G}_2$ such that $\tilde{\phi}(\tilde{K}_1) = \tilde{K}_2$, and hence it induces a diffeomorphism $\phi : \tilde{G}_1/\tilde{K}_1 \to \tilde{G}_2/\tilde{K}_2$ that preserves the symmetries, i.e.

$$s_{1_x} = \phi \circ s_{2_{\phi^{-1}(x)}} \circ \phi^{-1} \quad \text{for all } x \text{ in } \tilde{G}_1/\tilde{K}_1.$$

Thus the pairs $(\mathfrak{g}_1,\sigma_1)$ and $(\mathfrak{g}_2,\sigma_2)$ induce the same class of 4-symmetric spaces - diffeomorphic underlying differentiable manifolds. Hence, we consider two pairs $(\mathfrak{g}_1,\sigma_1)$, $(\mathfrak{g}_2,\sigma_2)$ as equivalent whenever there exists an isomorphism $\phi$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ which makes the above diagram commutative. Conversely, it is straightforward to see that if there exists an isometry between two Riemannian 4-symmetric space $(M_1,g_1,s_1)$, $(M_2,g_2,s_2)$ that preserves the symmetries, in the above sense, then it induces an isomorphism at Lie group level between the symmetry groups $G_1$ and $G_2$. Furthermore, the induced automorphism between the Lie algebras makes the above diagram commutative. i.e. $(\mathfrak{g}_1,\sigma_1)$ and $(\mathfrak{g}_2,\sigma_2)$ are equivalent.

## (a) Inner Automorphisms of Order Four of Compact Simple Lie Algebras

We shall classify the pairs $(\mathfrak{g},\sigma)$ up to equivalence. Reduction to the case $\mathfrak{g}$ simple is straightforward since any inner automorphism leaves invariant the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n$ of $\mathfrak{g}$ into its simple ideals. Thus we classify the inner automorphisms of order four of compact simple Lie algebras.

In what follows $\mathfrak{g}$ denotes a compact simple Lie algebra and $G$ the compact with trivial center Lie group whose Lie algebra is $\mathfrak{g}$. The group of inner automorphisms of $\mathfrak{g}$, $\mathrm{Int}(\mathfrak{g})$, is the subgroup of $GL(\mathfrak{g})$ whose Lie algebra is $ad(\mathfrak{g})$, the image under the adjoint representation of $\mathfrak{g}$ into $\mathfrak{gl}(\mathfrak{g})$. By definition, $\mathfrak{g}$ compact means $Int(\mathfrak{g})$ compact in the usual topological sense. $\mathfrak{g}$ semisimple implies that $ad(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$ and also that it coincides with the algebra $\partial(\mathfrak{g})$ of derivations of $\mathfrak{g}$. Since $\partial(\mathfrak{g})$ is the Lie algebra of the group of automorphisms of $\mathfrak{g}$, $Aut(\mathfrak{g})$, it follows that $Int(\mathfrak{g})$ is the identity component of $Aut(\mathfrak{g})$. It is also true that $Aut(\mathfrak{g})$ is compact, a result that follows from the fact that $Aut(\mathfrak{g})$ is a closed subgroup of the orthogonal group $O(\mathfrak{g})$ with respect to the Killing-Cartan form which is negative definite. In particular we have that the quotient group $Aut(\mathfrak{g})/Int(\mathfrak{g})$ is finite.

Another way of describing the group of inner automorphisms is by means of the adjoint representation, we have $Int(\mathfrak{g}) = Ad(G)$. By connectedness, the exponential map $\exp : \mathfrak{g} \to G$ is surjective, therefore any inner automorphism $\theta$ can be written in the form $\theta = Ad(\exp x)$ for some $x$ in $\mathfrak{g}$. Hence, the problem of classification of inner automorphisms of order four is equivalent to classifying the elements $x$ in $\mathfrak{g}$ such that $Ad(\exp x)$ is of order four.

Since the classification is up to conjugation, an important reduction of the problem is attained as follows by taking a maximal torus $T$ in $G$. Recall that if $\mathfrak{t}_0$ is the Lie algebra of $T$, then $\mathfrak{g}$ is the union $\cup_{g \in G} Ad(g)\mathfrak{t}_0$. This implies that every element in $G$ is of the form $\exp Ad(g)H$ for some $g$ in $G$ and some $H$ in $\mathfrak{t}_0$. But

$\exp \mathrm{Ad}(g)H = g(\exp H)g^{-1}$, and taking the adjoint representation: $\mathrm{Ad}(\exp \mathrm{Ad}(g)H) = \mathrm{Ad}(g)\mathrm{Ad}(\exp H)\mathrm{Ad}(g^{-1})$, i.e. every inner automorphism is conjugate - within the group of inner automorphisms—to an element in $\mathrm{Ad}_G(T)$, the image of the maximal torus $T$ under the adjoint representation of $G$ into $\mathrm{Int}(\mathfrak{g})$. Therefore attention can be restricted to the elements in $\mathfrak{t}_0$.

The algebra $\mathfrak{t}_0$ is a maximal abelian subalgebra in $\mathfrak{g}$, then when complexifying, $\mathfrak{t} = \mathfrak{t}_0^{\mathbb{C}}$ becomes a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

An additional restriction can be obtained by introducing the Weyl group with respect to $\mathfrak{t}_0$. The Weyl group is defined to be the group of inner automorphisms that preserves $\mathfrak{t}_0$. $W$ is isomorphic to the quotient $NT/T$ where $NT$ denotes the normalizer of $T$ in $G$. Both $NT$ and $T$ have the same Lie algebra $\mathfrak{t}_0$, and $NT$ being closed in $G$ is compact, thus $NT/T$ is finite. We consider the orbit space $W\backslash\mathfrak{t}_0$: If two elements $H_1$ and $H_2$ in $\mathfrak{t}_0$ belong to the same orbit, then there exists an element $t \in T$ such that $\mathrm{Ad}(t)H_1 = H_2$, hence $\mathrm{Ad}(\exp H_1)$ is conjugate to $\mathrm{Ad}(\exp H_2)$ by means of the inner automorphism $\mathrm{Ad}(t)$, i.e. two elements on the same orbit induce equivalent automorphisms. The description of $W\backslash\mathfrak{t}_0$ is given in terms of the Weyl chambers. For this, we let $\mathfrak{t}$ be the complexification of $\mathfrak{t}_0$, since $\mathfrak{t}_0$ is a maximal abelian subalgebra of $\mathfrak{g}$, $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, the complexification of $\mathfrak{g}$. Let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t} + \sum_{\alpha \in D} \mathfrak{g}^{\alpha} \quad \text{(direct sum)}$$

be the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}$. Here $\Delta$ denotes the root system. We denote by $\psi = \{\alpha_1, \ldots, \alpha_n\}$ a system of simple roots, and by $\mu = \sum m_i \alpha_i$ the maximal root. A Weyl chamber is defined to be a connected component of the subset of $\mathfrak{t}_0$ where all the members of $\psi$ are different from zero. Then we have that the orbit space $W \backslash \mathfrak{t}_0$ can be identified with the closure of any one Weyl chamber. I.e., more precisely, it is true that if $C$ is a Weyl chamber, then each orbit intersects $\overline{C}$ in one and only one point. Therefore every element in $\mathfrak{t}_0$ is conjugate to an element in $\overline{C}$.

$\overline{C}$ is still too big for our purposes, therefore we make a further reduction as follows: In $\mathfrak{t}_0$ we define the diagram of $\mathfrak{g}$ to be the set

$$D(\mathfrak{g}) = \{H \in \mathfrak{t}_0 : \alpha(H) \in 2\pi\sqrt{-1}\,\mathbb{Z} \text{ for some } \alpha \in \Delta\}.$$

Let $\mathfrak{t}_r = \mathfrak{t}_0 - D(\mathfrak{g})$, and let $P_0$ be a connected component of $\mathfrak{t}_r$ whose closure $\overline{P}_0$ contains the origin.

Let $\Gamma_\Delta$ be the group of affine transformations of $\mathfrak{t}_0$ generated by the reflections in the walls of $P_0$, then the orbit space $\mathfrak{t}_0/\Gamma_\Delta$ is equivalent to $\overline{P}_0$. Since the group $\Gamma_\Delta$ can be described as the semidirect product of $W$ and a sublattice $\mathfrak{t}_\Delta$ of the unit lattice $\mathfrak{t}_e = \{H \in \mathfrak{t}_0 : \exp H = e\}$, it follows that every element in $G$ is conjugate to an element in $\exp(\overline{P}_0)$.

If we take the Weyl chamber $C$ to satisfy the condition that $\alpha_i$ be positive on $-\sqrt{-1}\, C$ for all $i$, and take $P_0 \subset C$, then

$$-\sqrt{-1}\, P_0 = \{H \in -\sqrt{-1}\, \mathfrak{t}_0 : \alpha_i(H) \geq 0,\ i = 1,\ldots,n, \text{ and } \mu(H) \leq 2\pi\}.$$

We put $\mathfrak{D}_0 = \dfrac{1}{2\pi\sqrt{-1}} P_0$. $\mathfrak{D}_0$ is a simplex in $\sqrt{-1}\, \mathfrak{t}_0$ and it is defined as

$$\mathfrak{D}_0 = \{x \in \sqrt{-1}\, \mathfrak{t}_0 : \alpha_i(x) \geq 0,\ i = 1,\ldots,n,\ \mu(x) \leq 1\}$$

and it has vertices $\{v_0, v_1, \ldots, v_n\}$ given by

$$v_0 = 0, \quad \alpha_i(v_j) = \frac{1}{m_i}\delta_{ij}.$$

Furthermore, every element in $G$ is conjugate to an element of $\exp(2\pi\sqrt{-1}\, \mathfrak{D}_0)$. Thus we can restrict our attention to $\mathcal{D}_0$.

Now, let $x \in \mathfrak{D}_0$, $g = \exp(2\pi\sqrt{-1}\, x)$ and $\theta = \mathrm{Ad}(g)$, when is $\theta$ of order four? I.e., when is it true that $\theta^2 \neq \mathrm{id}$ but $\theta^4 = \mathrm{id}$. Since the adjoint representation gives an isomorphism between $G$ and $\mathrm{Int}(\mathfrak{g})$, this is equivalent to ask $g^2 \neq e$ and $g^4 = e$. For this to be the case, we ought to have $\alpha_i(2x) \notin \mathbb{Z}$ for some $i$, and $\alpha_i(4x) \in \mathbb{Z}$ for all $i$.

Write $4x = \sum_{i=1}^n a_i v_i$, then $n_i = \alpha_i(4x) = a_i/m_i \in \mathbb{Z}$, and $a_i = n_i m_i$, thus $x = \sum_{i=1}^n \frac{n_i m_i}{4} v_i$ and we must have that some $n_i$ is not even. The following basic proposition (contained in W. GI[28]) gives a normalization of the $x$'s so as to restrict the range of the coefficients $n_i m_i$ and thus it will lead toward the classification.

Proposition 1. Let $x \in \frac{1}{2\pi\sqrt{-1}} t_0$, and $k > 0$ be an integer such that $x \notin \frac{1}{2\pi\sqrt{-1}} t_e$ but $kx \in \frac{1}{2\pi\sqrt{-1}} t_e$, and replace by a transform $w(x) + \gamma \in \mathfrak{D}_0$, with $w \in W$ and $\gamma \in \frac{1}{2\pi\sqrt{-1}} t_e$, and decompose $x = \sum \frac{n_i m_i}{k} \nu_i$, where $n_i = \alpha_i(kx) \in \mathbb{Z}$.

Make this transformation in such a way as to minimize $\sum_{i=1}^{n} m_i n_i$. Then

(i) $0 \le n_i < k$ and $0 < n_i m_i \le k$, and $\sum n_i m_i = k$ implies that $m_j > 1$ whenever $n_j \neq 0$.

(ii) $n_j \le k/2$ if $m_j = 1$; and

(iii) the sets $I_t = \{i : n_i \ge t\}$ have cardinality $|I_1| \ge 1$ and $|I_t| < k/t$, and $I_t$ is empty for $t > k/2$.

Using this normalization we now list all the possibilities for inner automorphisms of order four.

Proposition 2. Let $\theta$ be an inner automorphism of order four on a compact or complex simple Lie algebra $\mathfrak{g}$. Then $\theta$ is conjugate, by an inner automorphism of $\mathfrak{g}$, to $\mathrm{Ad}(\exp 2\pi\sqrt{-1}\, x)$ where $x$ is as given below.

(i) $x = \frac{1}{4}\nu_j$ with $m_j = 1$.

(ii) $x = \frac{1}{4}(\nu_i+\nu_j)$ with $m_i = m_j = 1$

$x = \frac{1}{2}\nu_j$ with $m_j = 2$.

(iii) $x = \frac{1}{4}(\nu_i+\nu_j+\nu_k)$ with $m_i = m_j = m_k = 1$

$x = \frac{1}{4}(2\nu_i+\nu_j)$ with $m_i = 2,\ m_j = 1$.

$x = \frac{3}{4}\nu_j$ with $m_j = 3$

$x = \frac{1}{4}(2\nu_i+\nu_j)$ with $m_i = m_j = 1$.

(iv) $x = \frac{1}{2}(\nu_i+\nu_j)$ with $m_i = m_j = 2$

$x = \nu_i$ with $m_i = 4$.

Proof. The cases are listed so as to have

(i) $\mu(4x) = 1$ or $\sum n_i m_i = 1$, which clearly has the only solution given above

$$n_i = m_i = 1 \qquad x = \frac{1}{4}\nu_i.$$

(ii) $\mu(4x) = 2$ or $\sum n_i m_i = 2$, here we have $n_i = n_j = m_i = m_j = 1$ which gives one solution, also $n_j = 1$, $m_j = 2$ is another solution giving rise to $x = \frac{1}{2}\nu_j$ as required, however, the solution $n_j = 2,\ m_j = 1$ is ruled out since $n_i$ should not be divisible by two as remarked above, or also we would have

$x = \frac{1}{2} \nu_j$ and $2x = \nu_j \in \frac{1}{2\pi\sqrt{-1}} t_e$ and $x$ would induce an automorphism of order two.

(iii) $\mu(4x) = 3$; $\sum n_i m_i = 3$. The above solutions are clearly valid, one more solution could be $n_i = 3$, $m_i = 1$, i.e. $x = \frac{3}{4} \nu_i$ but by normalization, $n_i \le \frac{4}{2} = 2$, thus it is ruled out.

(iv) $\mu(4x) = 4$; $\sum n_i m_i = 4$. We cannot have $x = \frac{1}{4}(\nu_i + \nu_j + \nu_k + \nu_\ell)$ with $m_i = m_j = m_k = m_\ell = 1$ by normalization (see (iii) in the normalization ). Neither can we have $x = \nu_j$ with $n_i = m_i = 2$. This would give an automorphism of order two. The same applies to $x = \frac{1}{2}(\nu_i + \nu_j)$ with $n_i = n_j = 2$, $m_i = m_j = 1$. ///

As for the fixed point set of an inner automorphism, a root system can be given in terms of the roots of $\mathfrak{g}$. The following proposition is also in W.G.I. [28].

<u>Proposition 3.</u> Let $\mathfrak{g}$ be a compact simple Lie algebra with simple root system $\psi = \{\alpha_1, \ldots, \alpha_n\}$, and $\theta$ an inner automorphism of $\mathfrak{g}$. Normalize $x$, so $\theta = \mathrm{Ad}(\exp 2\pi\sqrt{-1}\, x)$ where $x = \sum_{i=1}^{n} c_i \nu_i \in \mathfrak{D}_0$. If $\mu$ is the maximal root of $\mathfrak{g}$ so that $\mu(x) = \sum c_i$, then $\psi_x$, defined as follows, is a simple root system for $\mathfrak{g}^\theta$, the fixed point algebra of $\theta$:

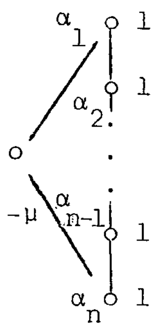
$$\psi_x = \{\alpha_i \in \psi : c_i = 0\}, \qquad \text{if} \quad \mu(x) < 1$$

$$\psi_x = \{\alpha_i \in \psi : c_i = 0\} \cup \{-\mu\}, \quad \text{if} \quad \mu(x) = 1.$$

Using these last two propositions, we draw the following table. It gives a complete list of the possibilities for $x$ an element in $\sqrt{-1}\, \mathfrak{t}_0$ inducing an automorphism of order four $\theta = \mathrm{Ad}(\exp 2\pi\sqrt{-1}\, x)$, the fixed point algebra $\mathfrak{g}^\theta$, and the simple root system $\psi_x$ of $\mathfrak{g}^\theta$.

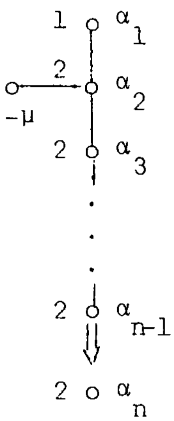
Here $\sim$ denotes equivalence in the full group of automorphisms.

TABLE I

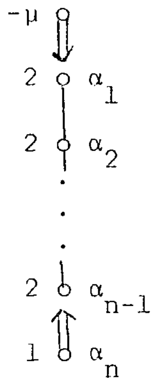
| $\mathfrak{g}$ | $x$ | $\psi_x$ | $\mathfrak{g}^{,\theta}$ |
|---|---|---|---|
| $\mathfrak{a}_1$ | $\frac{1}{4}\nu_1$ | empty | $\mathfrak{T}^1$ |
| $\mathfrak{a}_n,\ n \geq 2$ | $\frac{1}{4}\nu_i[\sim\frac{1}{4}\nu_{n-i+1}]$ | $\{\alpha_1,\ldots,\alpha_{i-1},\alpha_{i+1},\ldots,\alpha_n\}$ | $\mathfrak{a}_{i-1}\oplus\mathfrak{a}_{n-i}\oplus\mathfrak{T}^1$ |
|  | $\frac{1}{4}(\nu_i+\nu_j),(i<j)$ $[\sim\frac{1}{4}(\nu_r+\nu_s)$ if $\{r-1,s-r-1,n-s\} = \{i-1,j-i-1,n-j\}]$ | $\{\alpha_1,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_{j-1};$ $\alpha_{j+1},\ldots,\alpha_n\}$ | $\mathfrak{a}_{i-1}\oplus\mathfrak{a}_{j-i-1}$ $\oplus\mathfrak{a}_{n-j}\oplus\mathfrak{T}^2$ |
| | $\frac{1}{4}(\nu_i+\nu_j+\nu_k)$, $i<j<k$ $[\sim\frac{1}{4}(\nu_r+\nu_s+\nu_t)$ if $\{r-1,s-r-1,t-s-1,$ $n-t\} = \{i-1,j-i-1,$ $k-j-1,n-k\}]$ | $\{\alpha_1,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_{j-1};$ $\alpha_{j+1},\ldots,\alpha_{k-1};$ $\alpha_{k+1},\ldots,\alpha_n\}$ | $\mathfrak{a}_{i-1}\oplus\mathfrak{a}_{j-i-1}$ $\oplus\mathfrak{a}_{k-j-1}$ $\oplus\mathfrak{a}_{n-k}\oplus\mathfrak{T}^3$ |
| | $\frac{1}{4}(2\nu_i+\nu_j),i<j$ $[\sim\frac{1}{4}(2\nu_r+\nu_s)$ if $i-1 = n-r$ $j-1 = n-s]$ | $\{\alpha_1,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_{j-1};$ $\alpha_{j+1},\ldots,\alpha_n\}$ | $\mathfrak{a}_{i-1}\oplus\mathfrak{a}_{j-i-1}$ $\oplus\mathfrak{a}_{n-j}\oplus\mathfrak{T}^2$ |

<u>Remark</u>. For a global formulation see below.

TABLE I (cont.)

| $\mathfrak{g}$ | $x$ | $\psi_x$ | $\mathfrak{g}^\theta$ |
|---|---|---|---|
| $\mathfrak{b}_n$, $n \geq 2$ | $\frac{1}{4}\nu_1$ | $\{\alpha_2,\ldots,\alpha_n\}$ | $\mathfrak{b}_{n-1} \oplus \mathfrak{T}^1$ |
|  | $\frac{1}{2}\nu_i (2\leq i\leq n)$ | $\{\alpha_1,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_n\}$ | $\mathfrak{a}_{i-1} \oplus \mathfrak{b}_{n-i} \oplus \mathfrak{T}^1$ |
| | $\frac{1}{4}(\nu_1+2\nu_i)(2\leq i\leq n)$ | $\{\alpha_2,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_n\}$ | $\mathfrak{a}_{i-2} \oplus \mathfrak{b}_{n-i} \oplus \mathfrak{T}^2$ |
| | $\frac{1}{2}(\nu_i+\nu_j)(2\leq i<j\leq n)$ | $\{\alpha_1,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_{j-1};$ $\alpha_{j+1},\ldots,\alpha_n\}\cup\{-\mu\}$ | $\mathfrak{d}_i \oplus \mathfrak{a}_{j-i-1}$ $\oplus \mathfrak{b}_{n-j} \oplus \mathfrak{T}^1$ |

<u>Remark</u>. A global formulation is given below.

| $\mathfrak{g}$ | $x$ | $\psi_x$ | $\mathfrak{g}^\theta$ |
|---|---|---|---|
| $\mathfrak{c}_n$, $n \geq 3$ | $\frac{1}{4}\nu_n$ | $\{\alpha_1,\ldots,\alpha_{n-1}\}$ | $\mathfrak{a}_{n-1} \oplus \mathfrak{T}^1$ |
| <br> | $\frac{1}{2}\nu_i (1 \leq i \leq n-1)$ | $\{\alpha_1,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_n\}$ | $\mathfrak{a}_{i-1} \oplus \mathfrak{c}_{n-i} \oplus \mathfrak{T}^1$ |
| | $\frac{1}{4}(\nu_n+2\nu_i)(1 \leq i \leq n-1)$ | $\{\alpha_1,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_{n-1}\}$ | $\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{n-i-1} \oplus \mathfrak{T}^2$ |
| | $\frac{1}{2}(\nu_i+\nu_j)(1 \leq i < j \leq n-1)$ | $\{\alpha_1,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_{j-1};$ $\alpha_{j+1},\ldots,\alpha_n\} \cup \{-\mu\}$ | $\mathfrak{c}_i \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{c}_{n-j} \oplus \mathfrak{T}^1$ |

Remark. See below for a global formulation.

TABLE I (cont.)

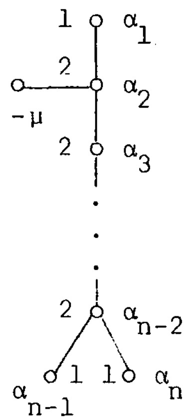
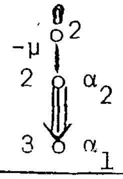
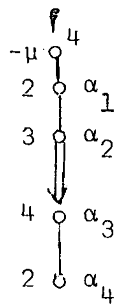
| $\mathfrak{g}$ | x | $\psi_x$ | $\mathfrak{g}^{\theta}$ |
|---|---|---|---|
| $\mathfrak{d}_n$, $n \geq 4$   | $\frac{1}{4}\nu_1$ | $\{\alpha_2,\ldots,\alpha_n\}$ | $\mathfrak{d}_{n-1} \oplus \mathfrak{T}^1$ |
| | $\frac{1}{4}(\nu_n)[\sim\frac{1}{4}\nu_{n-1}]$ | $\{\alpha_1,\ldots,\alpha_{n-1}\}$ | $\mathfrak{a}_{n-1} \oplus \mathfrak{T}^1$ |
| | $\frac{1}{4}(\nu_1+\nu_n)[\sim\frac{1}{4}(\nu_1+\nu_{n-1})]$ | $\{\alpha_2,\ldots,\alpha_{n-1}\}$ | $\mathfrak{a}_{n-2} \oplus \mathfrak{T}^2$ |
| | $\frac{1}{4}(\nu_{n-1}+\nu_n)$ | $\{\alpha_1,\ldots,\alpha_{n-2}\}$ | $\mathfrak{a}_{n-2} \oplus \mathfrak{T}^2$ |
| | $\frac{1}{2}\nu_i (2 \leq i \leq n-2)$ | $\{\alpha_1,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_n\}$ | $\mathfrak{a}_{i-1} \oplus \mathfrak{d}_{n-i} \oplus \mathfrak{T}^1$ |
| | | | |
| | $\frac{1}{4}(\nu_1+\nu_{n-1}+\nu_n)$ | $\{\alpha_2,\ldots,\alpha_{n-2}\}$ | $\mathfrak{a}_{n-3} \oplus \mathfrak{T}^3$ |
| | $\frac{1}{4}(\nu_1+2\nu_i)(2 \leq i \leq n-2)$ | $\{\alpha_2,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_n\}$ | $\mathfrak{a}_{i-2} \oplus \mathfrak{d}_{n-i} \oplus \mathfrak{T}^2$ |
| | | | |
| | $\frac{1}{4}(\nu_{n-1}+2\nu_i)$ $[\sim\frac{1}{4}(\nu_n+2\nu_i)$ $(2 \leq i \leq n-2)$ | $\{\alpha_1,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_{n-2},\alpha_n\}$ | $\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{n-i-1} \oplus \mathfrak{T}^2$ |
| | $\frac{1}{4}(2\nu_1+\nu_n)$ $[\sim\frac{1}{4}(2\nu_1+\nu_{n-1})]$ | $\{\alpha_2,\ldots,\alpha_{n-1}\}$ | $\mathfrak{a}_{n-2} \oplus \mathfrak{T}^2$ |

TABLE I (cont.)

| $\mathfrak{g}$ | $x$ | $\psi_x$ | $\mathfrak{g}^{\theta}$ |
|---|---|---|---|
| $\mathfrak{d}_n$ (continuation) | $\frac{1}{4}(2\nu_{n-1}+\nu_n)$ $[\sim\frac{1}{4}(2\nu_n+\nu_{n-1})]$ | $\{\alpha_1,\ldots,\alpha_{n-2}\}$ | $\mathfrak{a}_{n-2}\oplus\mathfrak{T}^2$ |
| | $\frac{1}{4}(2\nu_n+\nu_1)[\sim\frac{1}{4}(2\nu_{n-1}+\nu_1)$ | $\{\alpha_2,\ldots,\alpha_{n-1}\}$ | $\mathfrak{a}_{n-2}\oplus\mathfrak{T}^2$ |
| | $\frac{1}{2}(\nu_i+\nu_j)(2\leq i<j\leq n-2)$ | $\{\alpha_1,\ldots,\alpha_{i-1};$ $\alpha_{i+1},\ldots,\alpha_{j-1};$ $\alpha_{j+1},\ldots,\alpha_{\ell}\}\cup\{-\mu\}$ | $\mathfrak{d}_i\oplus\mathfrak{a}_{j-1-i}$ $\oplus\mathfrak{d}_{n-j}\oplus\mathfrak{T}^1$ |

<u>Remark</u>. A global formulation is obtained below for all the entries but the one corresponding to $(\mathfrak{d}_n,\mathfrak{a}_{n-3}\oplus\mathfrak{T}^3)$.

TABLE I (cont.)

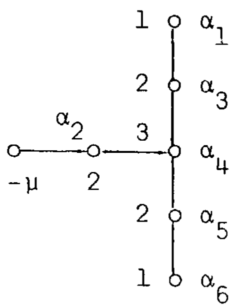
| $\mathfrak{g}$ | $x$ | $\psi_x$ | $\mathfrak{g}^{\theta}$ |
|---|---|---|---|
|  | $\frac{1}{2}\nu_2$ | $\{\alpha_1\}$ | $\mathfrak{a}_1 \oplus \mathfrak{T}^1$ |
| | $\frac{3}{4}\nu_1$ | $\{\alpha_2\}$ | $\mathfrak{a}_1 \oplus \mathfrak{T}^1$ |
|  | $\frac{1}{2}\nu_1$ | $\{\alpha_2,\alpha_3,\alpha_4\}$ | $\mathfrak{c}_3 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}\nu_4$ | $\{\alpha_1,\alpha_2,\alpha_3\}$ | $\mathfrak{b}_3 \oplus \mathfrak{T}^1$ |
| | $\frac{3}{4}\nu_2$ | $\{\alpha_1,\alpha_3,\alpha_4\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}(\nu_1+\nu_4)$ | $\{\alpha_2,\alpha_3\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{T}^1$ |
| | $\nu_3$ | $\{\alpha_1,\alpha_2;\alpha_4\}\cup\{-\mu\}$ | $\mathfrak{a}_3 \oplus \mathfrak{a}_1$ |

<u>Remarks</u>. Although the two given automorphisms of $\mathfrak{g}_2$ have isomorphic fixed point sets, they are not conjugate. Hence the corresponding compact simply connected 4-symmetric spaces are not isomorphic.

For $\mathfrak{f}_4$ we can state

<u>Proposition</u>. Two inner automorphisms of order four of $\mathfrak{f}_4$ with isomorphic fixed point sets are conjugate, (actually within the group of inner automorphisms).

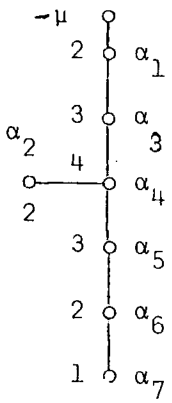
TABLE I (cont.)

| $\mathfrak{g}$ | x | $\psi_x$ | $\mathfrak{g}^{\theta}$ |
|---|---|---|---|
| $\mathfrak{e}_6$  | $\frac{1}{4}\nu_1[\sim\frac{1}{4}\nu_6]$ | $\{\alpha_2,\ldots,\alpha_6\}$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{4}(\nu_1+\nu_6)$ | $\{\alpha_2,\ldots,\alpha_5\}$ | $\mathfrak{d}_4 \oplus \mathfrak{T}^2$ |
| | $\frac{1}{2}\nu_2$ | $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$ | $\mathfrak{a}_5 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}\nu_3[\sim\frac{1}{2}\nu_5]$ | $\{\alpha_1,;\alpha_2,\alpha_4,\alpha_5,\alpha_6\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_4 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{4}(\nu_1+2\nu_3)$ $[\sim\frac{1}{4}(\nu_6+2\nu_5)]$ | $\{\alpha_2,\alpha_4,\alpha_5,\alpha_6\}$ | $\mathfrak{a}_4 \oplus \mathfrak{T}^2$ |
| | $\frac{1}{4}(\nu_1+2\nu_5)$ $[\sim\frac{1}{4}(\nu_6+2\nu_3)]$ | $\{\alpha_2,\alpha_3,\alpha_4,\alpha_6\}$ | $\mathfrak{a}_3 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^2$ |
| | $\frac{1}{4}(\nu_1+2\nu_2)$ $[\sim\frac{1}{4}(\nu_6+2\nu_2)]$ | $\{\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$ | $\mathfrak{a}_4 \oplus \mathfrak{T}^2$ |
| | $\frac{3}{4}\nu_4$ | $\{\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6\}$ | $\mathfrak{a}_2 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{4}(2\nu_1+\nu_6)$ $[\sim\frac{1}{4}(2\nu_6+\nu_1)]$ | $\{\alpha_2,\ldots,\alpha_5\}$ | $\mathfrak{d}_4 \oplus \mathfrak{T}^2$ |
| | $\frac{1}{2}(\nu_3+\nu_5)$ | $\{\alpha_1,\alpha_2,\alpha_4,\alpha_6\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_3 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}(\nu_2+\nu_3)$ $[\sim\frac{1}{2}(\nu_2+\nu_5)]$ | $\{\alpha_1,\alpha_4,\alpha_5,\alpha_6\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_3 \oplus \mathfrak{T}^1$ |

Remark. The following is true.

Proposition. Two inner automorphisms of $\mathfrak{e}_6$ with isomorphic fixed sets are conjugate.

TABLE I (cont.)

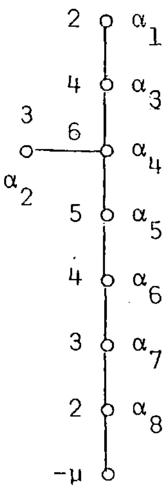
| $\mathfrak{g}$ | x | $\psi_x$ | $\mathfrak{g}^{\theta}$ |
|---|---|---|---|
| $\mathfrak{e}_7$  | $\frac{1}{4}\nu_7$ | $\{\alpha_1,\ldots,\alpha_6\}$ | $\mathfrak{e}_6 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}\nu_1$ | $\{\alpha_2,\ldots,\alpha_7\}$ | $\mathfrak{d}_6 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}\nu_6$ | $\{\alpha_1,\ldots,\alpha_5;\alpha_7\}$ | $\mathfrak{d}_5 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}\nu_2$ | $\{\alpha_1,\alpha_3,\ldots,\alpha_7\}$ | $\mathfrak{a}_6 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{4}(\nu_7+2\nu_1)$ | $\{\alpha_2,\ldots,\alpha_6\}$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^2$ |
| | $\frac{1}{4}(\nu_7+2\nu_6)$ | $\{\alpha_1,\ldots,\alpha_5\}$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^2$ |
| | $\frac{1}{4}(\nu_7+2\nu_2)$ | $\{\alpha_1,\alpha_3,\ldots,\alpha_6\}$ | $\mathfrak{a}_5 \oplus \mathfrak{T}^2$ |
| | $\frac{3}{4}\nu_3$ | $\{\alpha_1;\alpha_2,\alpha_4,\ldots,\alpha_6,\alpha_7\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_5 \oplus \mathfrak{T}^1$ |
| | $\frac{3}{4}\nu_5$ | $\{\alpha_1,\ldots,\alpha_4;\alpha_6,\alpha_7\}$ | $\mathfrak{a}_4 \oplus \mathfrak{a}_2 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}(\nu_1+\nu_6)$ | $\{\alpha_2,\ldots,\alpha_5;\alpha_7\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{d}_4 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}(\nu_1+\nu_2)$ | $\{\alpha_3,\ldots,\alpha_7\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_5 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}(\nu_2+\nu_6)$ | $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_7\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_5 \oplus \mathfrak{T}^1$ |
| | $\nu_4$ | $\{\alpha_1,\alpha_2,\alpha_3;\alpha_5,\alpha_6,\alpha_7\}\cup\{-\mu\}$ | $\mathfrak{a}_3 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_3$ |

Remarks. The following is true.

Proposition. There are two nonconjugate classes of inner automorphisms of order four of $\mathfrak{e}_7$ whose fixed point sets are isomorphic to $\mathfrak{a}_1 \oplus \mathfrak{a}_5 \oplus \mathfrak{T}^1$. With this exception, any two inner automorphisms of order four of $\mathfrak{e}_6$ with isomorphic fixed point sets are conjugate.

Comment. The geometric difference between the two options for $\mathfrak{a}_1 \oplus \mathfrak{a}_5 \oplus \mathfrak{T}^1$ is the existence of invariant almost complex structures (see Section 9).

TABLE I (cont.)

| $\mathfrak{g}$ | x | $\psi_x$ | $\mathfrak{g}^\theta$ |
|---|---|---|---|
| $\mathfrak{e}_8$ | $\frac{1}{2}\nu_1$ | $\{\alpha_2,\ldots,\alpha_8\}$ | $\mathfrak{d}_7 \oplus \mathfrak{T}^1$ |
|  | $\frac{1}{2}\nu_8$ | $\{\alpha_1,\ldots,\alpha_7\}$ | $\mathfrak{e}_7 \oplus \mathfrak{T}^1$ |
| | $\frac{3}{4}\nu_2$ | $\{\alpha_1,\alpha_3,\ldots,\alpha_8\}$ | $\mathfrak{a}_7 \oplus \mathfrak{T}^1$ |
| | $\frac{3}{4}\nu_7$ | $\{\alpha_1,\ldots,\alpha_6;\alpha_8\}$ | $\mathfrak{e}_6 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}(\nu_1+\nu_8)$ | $\{\alpha_2,\ldots,\alpha_7\}\cup\{-\mu\}$ | $\mathfrak{d}_6 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^1$ |
| | $\nu_3$ | $\{\alpha_1;\alpha_2,\alpha_4,\ldots,\alpha_8\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_7$ |
| | $\nu_6$ | $\{\alpha_1,\ldots,\alpha_5;\alpha_7,\alpha_8\}\cup\{-\mu\}$ | $\mathfrak{d}_5 \oplus \mathfrak{a}_3$ |

Remark. The following is true.

Proposition. Any two inner automorphisms of order four of $\mathfrak{e}_8$ with isomorphic fixed point sets are conjugate.

Description of the base space and the universal cover of the fiber

The next step in the classification of compact Riemannian 4-symmetric spaces is to obtain the global formulation of the above tables. I.e. we should draw a table of all the compact simply connected coset spaces $M = G/K$ where $G$ is a compact connected Lie group acting effectively, $\mathfrak{k} = \mathfrak{g}^{\theta}$ the Lie algebra of $K$ and $\theta$ an inner automorphism of order four. We shall only do this for the classical Lie algebras (see the end of this section).

We shall conclude this section with an application of the local classification to obtain some more information of the geometry of these spaces. Recall that in Section 4 we saw that a compact Riemannian 4-symmetric space fibers over a compact Cartan symmetric space with fiber a compact Cartan symmetric space. Furthermore, this fibration can be regarded as a Riemannian submersion. Thus, if $M = G/K$ is the compact simply connected Riemannian 4-symmetric space corresponding to one of the entries in the above tables, it is interesting to describe the fiber and the base space. For example one could ask whether or not the fiber is an Hermitian 2-symmetric space. The answer to this question is equivalent to answering whether or not the total space admits an almost complex structure invariant under the symmetries. These structures will be studied in Section 9. Here we will describe the base and the fiber in general.

Let $(\mathfrak{g}, \mathfrak{g}^{\theta}, \theta)$ be an entry in the tables, let $\tilde{G}$ be the compact simply connected Lie group associated with $\mathfrak{g}$, and let $\tilde{K}$ be the connected subgroup of $\tilde{G}$ corresponding to $\mathfrak{g}^{\theta}$, then $\tilde{G}/\tilde{K}$ is the simply connected 4-symmetric space associated with $(\mathfrak{g},\mathfrak{g}^{\theta},\theta)$. (Note that $\tilde{G}$ does not necessarily act effectively on $\tilde{G}/\tilde{K}$). Let $\theta$ (same letter) be the induced automorphism on $\tilde{G}$, then the fixed point set $\tilde{G}^{\theta^2}$ of $\theta^2$ is a compact connected subgroup and both $\tilde{G}/\tilde{G}^{\theta^2}$ and $\tilde{G}^{\theta^2}/\tilde{K}$ are compact Cartan symmetric spaces and the diagram

$$\begin{array}{ccc} \tilde{G}^{\theta^2}/\tilde{K} & \hookrightarrow & \tilde{G}/K \\ & & \downarrow \\ & & \tilde{G}/\tilde{G}^{\theta^2} \end{array}$$

gives the fibration of $\tilde{G}/\tilde{K}$. Thus the way we proceed to give the information of the fiber $F = \tilde{G}^{\theta^2}/\tilde{K}$ and the base space $B = \tilde{G}/\tilde{G}^{\theta^2}$ is by first describing the fixed point set $\mathfrak{g}^{\theta^2}$ of $\theta^2$ in our above tables and then we find out for $(\mathfrak{g},\mathfrak{g}^{\theta^2})$ which is the compact simply connected Cartan symmetric space associated with it. This space will be the base space B. We do the same for $(\mathfrak{g}^{\theta^2},\mathfrak{g}^{\theta})$. The space associated with this pair will be the universal cover of the fiber F.

Proposition 3 gives a description for $\mathfrak{g}^{\theta}$, $\theta = \mathrm{Ad}(\exp 2\pi\sqrt{-1}\,x)$, $x \in i\mathfrak{t}_0$ whenever $\mu(x) \le 1$, $\alpha_i(x) \ge 0$. Thus this proposition could be applied to our present situation whenever $2x \in \mathfrak{D}_0$, where $x$ is as given in Proposition 2. However, this is not always the case. The proof of this proposition yields the result that $2x \in \mathfrak{D}_0$ if and only if $x$ is of one of the following forms:

$$x = \frac{1}{4}\nu_j \quad \text{with} \quad m_j = 1$$

$$x = \frac{1}{4}(\nu_i+\nu_j) \quad \text{with} \quad m_i = m_j = 1$$

$$x = \frac{1}{2}\nu_j \quad \text{with} \quad m_j = 2.$$

Hence, for these three cases we obtain an explicit description of $\mathfrak{g}^{\theta^2}$. The following tables give this information. The first table explains how to construct a root system $\psi_{2x}$ for $\mathfrak{g}^{\theta^2}$ out of a root system $\psi$ for $\mathfrak{g}$. It also says whether or not $\mathfrak{g}^{\theta^2}$ is the centralizer of a toral algebra. (For this we use Proposition 2 of Section 9 below). In the second table we write $x$ and $\mathfrak{g}^{\theta}$ as given in our tables above, and instead of giving the root system $\psi_x$ of $\mathfrak{g}^{\theta}$, we now write $\mathfrak{g}^{\theta^2}$.

For equivalences amongst the different inner autormophisms and for notation and the extended Dynkin diagram, we refer the reader to TABLE I.

| $x$ | $\psi_x$ | $\psi_{2x}$ | |
|---|---|---|---|
| $\frac{1}{4}\nu_j, m_j=1$ | $\{\alpha_i \in \psi : c_i = 0\}$ | $\{\alpha_i \in \psi : c_i = 0\}$ | $\mathfrak{g}^{\theta^2}$ is the centralizer of a toral algebra |
| $\frac{1}{4}(\nu_i+\nu_j)$, $m_i=m_j=1$ | $\{\alpha_i \in \psi : c_i = 0\}$ | $\{\alpha_i \in \psi : c_i = 0\} \cup \{-\mu\}$ | $\mathfrak{g}^{\theta^2}$ is the centralizer of a toral algebra |
| $\frac{1}{2}\nu_j$, $m_j=2$ | $\{\alpha_i \in \psi : c_i = 0\}$ | $\{\alpha_i \in \psi : c_i = 0\} \cup \{-\mu\}$ | $\mathfrak{g}^{\theta^2}$ is not the centralizer of a toral algebra |

Remarks. In Section 9 it is shown that for these three cases $\mathfrak{g}^{\theta}$ is the centralizer of a toral algebra, and hence that the corresponding spaces are almost Hermitian 4-symmetric. The above table then shows that for the first two cases, the base space is an Hermitian 2-symmetric space, whereas for the third this is not so. Of course, the fiber in all these cases is always Hermitian 2-symmetric.

Also, note that for the first case, $x = \frac{1}{4} \nu_j$, $m_j = 1$, $\mathfrak{g}^{\theta} = \mathfrak{g}^{\theta^2}$. This means that the squares of the corresponding symmetries are in fact the geodesic involutions and hence that the original space is Hermitian 2-symmetric.

| $\mathfrak{g}$ | $x$ | $\mathfrak{g}^{\theta}$ | $\mathfrak{g}^{\theta^2}$ |
|---|---|---|---|
| $\mathfrak{a}_1$ | $\frac{1}{4}\nu_1$ | $\mathfrak{T}^1$ | $\mathfrak{T}^1$ |
| $\mathfrak{a}_n, n\geq 2$ | $\frac{1}{4}\nu_i$ | $\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{n-i} \oplus \mathfrak{T}^1$ | $\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{n-i} \oplus \mathfrak{T}^1$ |
| | $\frac{1}{4}(\nu_i+\nu_j), (i<j)$ | $\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{a}_{n-j} \oplus \mathfrak{T}^2$ | $\mathfrak{a}_{n-j+i} \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{T}^1$ |
| $\mathfrak{b}_n, n\geq 2$ | $\frac{1}{4}\nu_1$ | $\mathfrak{b}_{n-1} \oplus \mathfrak{T}^1$ | $\mathfrak{b}_{n-1} \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}\nu_i (2\leq i\leq n)$ | $\mathfrak{a}_{i-1} \oplus \mathfrak{b}_{n-i} \oplus \mathfrak{T}^1$ | $\mathfrak{d}_i \oplus \mathfrak{b}_{n-i}$ |
| $\mathfrak{c}_n, n\geq 3$ | $\frac{1}{4}\nu_n$ | $\mathfrak{a}_{n-1} \oplus \mathfrak{T}^1$ | $\mathfrak{a}_{n-1} \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}\nu_i (1\leq i\leq n-1)$ | $\mathfrak{a}_{i-1} \oplus \mathfrak{c}_{n-i} \oplus \mathfrak{T}^1$ | $\mathfrak{c}_i \oplus \mathfrak{c}_{n-i}$ |

| $\mathfrak{g}$ | $x$ | $\mathfrak{g}^{\theta}$ | $\mathfrak{g}^{\theta^2}$ |
|---|---|---|---|
| $\mathfrak{d}_n, n\geq 4$ | $\frac{1}{4}\nu_1$ | $\mathfrak{d}_{n-1}\oplus\mathfrak{T}^1$ | $\mathfrak{d}_{n-1}\oplus\mathfrak{T}^1$ |
| | $\frac{1}{4}\nu_n[\sim\frac{1}{4}\nu_{n-1}]$ | $\mathfrak{a}_{n-1}\oplus\mathfrak{T}^1$ | $\mathfrak{a}_{n-1}\oplus\mathfrak{T}^1$ |
| | $\frac{1}{4}(\nu_1+\nu_n)[\sim\frac{1}{4}(\nu_1+\nu_{n-1})]$ | $\mathfrak{a}_{n-2}\oplus\mathfrak{T}^2$ | $\mathfrak{a}_{n-1}\oplus\mathfrak{T}^1$ |
| | $\frac{1}{4}(\nu_{n-1}+\nu_n)$ | $\mathfrak{a}_{n-2}\oplus\mathfrak{T}^2$ | $\mathfrak{d}_{n-1}\oplus\mathfrak{T}^1$ |
| | $\frac{1}{2}\nu_i(2\leq i\leq n-2)$ | $\mathfrak{a}_{i-1}\oplus\mathfrak{d}_{n-i}\oplus\mathfrak{T}^1$ | $\mathfrak{d}_i\oplus\mathfrak{d}_{n-i}$ |
| $\mathfrak{b}_2$ | $\frac{1}{2}\nu_2$ | $\mathfrak{a}_1\oplus\mathfrak{T}^1$ | $\mathfrak{a}_1\oplus\mathfrak{a}_1$ |
| $\mathfrak{f}_4$ | $\frac{1}{2}\nu_1$ | $\mathfrak{c}_3\oplus\mathfrak{T}^1$ | $\mathfrak{c}_3\oplus\mathfrak{a}_1$ |
| | $\frac{1}{2}\nu_4$ | $\mathfrak{b}_3\oplus\mathfrak{T}^1$ | $\mathfrak{b}_4$ |
| $\mathfrak{e}_6$ | $\frac{1}{4}\nu_1[\sim\frac{1}{4}\nu_6]$ | $\mathfrak{d}_5\oplus\mathfrak{T}^1$ | $\mathfrak{d}_5\oplus\mathfrak{T}^1$ |
| | $\frac{1}{4}(\nu_1+\nu_6)$ | $\mathfrak{d}_4\oplus\mathfrak{T}^2$ | $\mathfrak{d}_5\oplus\mathfrak{T}^1$ |
| | $\frac{1}{2}\nu_2$ | $\mathfrak{a}_5\oplus\mathfrak{T}^1$ | $\mathfrak{a}_5\oplus\mathfrak{a}_1$ |
| | $\frac{1}{2}\nu_3[\sim\frac{1}{2}\nu_5]$ | $\mathfrak{a}_1\oplus\mathfrak{a}_4\oplus\mathfrak{T}^1$ | $\mathfrak{a}_5\oplus\mathfrak{a}_1$ |
| $\mathfrak{e}_7$ | $\frac{1}{4}\nu_7$ | $\mathfrak{e}_6\oplus\mathfrak{T}^1$ | $\mathfrak{e}_6\oplus\mathfrak{T}^1$ |
| | $\frac{1}{2}\nu_1$ | $\mathfrak{d}_6\oplus\mathfrak{T}^1$ | $\mathfrak{d}_6\oplus\mathfrak{a}_1$ |
| | $\frac{1}{2}\nu_6$ | $\mathfrak{d}_5\oplus\mathfrak{a}_1\oplus\mathfrak{T}^1$ | $\mathfrak{a}_1\oplus\mathfrak{d}_6$ |
| | $\frac{1}{2}\nu_2$ | $\mathfrak{a}_6\oplus\mathfrak{T}^1$ | $\mathfrak{a}_7$ |

| $\mathfrak{g}$ | $x$ | $\mathfrak{g}^{\theta}$ | $\mathfrak{g}^{\theta^2}$ |
|---|---|---|---|
| $\mathfrak{e}_8$ | $\frac{1}{2}\nu_1$ | $\mathfrak{d}_7 \oplus \quad {}^{1}$ | $\mathfrak{d}_8$ |
| | $\frac{1}{2}\nu_8$ | $\mathfrak{e}_7 \oplus \quad {}^{1}$ | $\mathfrak{e}_7 \oplus \mathfrak{a}_1$ |

Remarks. It should be desirable to have a general algorithm which could work for the remaining cases of $x$. A thing one can do is to carry $x$, by means of reflections, into $\mathfrak{D}_0$. In this way we could obtain an equivalent automorphism of order four for which Proposition 3 could be applied. I have not yet found a general method to do this.

On the other hand, note that for all these cases it is now possible to identify the base space, and somehow, by inspection, to do the same for the fiber (i.e. the universal cover of the fiber). In what follows, we will do this. Actually we shall go beyond these three cases and shall give a complete description in all possible cases. We do it as follows.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ or a compact simple Lie algebra over $\mathbb{R}$, and let $\theta = \mathrm{Ad}(\exp 2\pi\sqrt{-1}\, x)$ be an inner automorphism of order four as given in the tables.

1. Study the inner involutions $\sigma : \mathfrak{g} \to \mathfrak{g}$ and describe their fixed point sets $\mathfrak{g}^{\sigma}$. Observe that if $\theta : \mathfrak{g} \to \mathfrak{g}$ is an inner automorphism of order four, then $\theta^2 : \mathfrak{g} \to \mathfrak{g}$ is an inner involution, hence $\mathfrak{g}^{\theta^2}$ must correspond with some $\mathfrak{g}^{\sigma}$.

2. Given $\mathfrak{g}^\sigma$ as in 1, study all inner involutions $\rho : \mathfrak{g}^\sigma \to \mathfrak{g}^\sigma$ and their fixed point sets $(\mathfrak{g}^\sigma)^\rho$.

3. Find out which sets $(\mathfrak{g}^\sigma)^\rho$ correspond with the Lie algebra $\mathfrak{g}^\theta$.

In most cases (see below for the few exceptions) the correspondence in 3 is one to one avoiding any possible ambiguity.

The simplest cases are when $\mathfrak{g}$ is an exceptional Lie algebra. We start with them.

The classification of the involutions of the simple Lie algebras over $\mathbb{C}$ is given in Tables II and III in Helgason [9] pp. 514-515. Actually, these tables only give the fixed point sets of the involutions, however if two involutions on a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ have isomorphic fixed point sets, they are conjugate. Since we shall be identifying the base space and the fiber, we shall be continuously referring to Table V in [9] pp. 518, where a complete list is given of the simply connected irreducible Riemannian 2-symmetric spaces. (Table IV p. 516 gives the Lie groups for the simple Lie algebra over $\mathbb{C}$ and their compact real forms).

$\underline{\mathfrak{g} = \mathfrak{g}_2}$

This simple Lie algebra has only one involution. This involution is inner and has as its fixed point set $\mathfrak{a}_1 \oplus \mathfrak{a}_1$. In particular this shows that the automorphism $\theta = \mathrm{Ad}(\exp 2\pi\sqrt{-1}\, x)$ with $x = \frac{1}{2}\nu_2$ or $x = \frac{3}{4}\nu_1$ as given in Table I do not give rise to 2-symmetric pairs, i.e. the couple $(\mathfrak{g}, \mathfrak{g}^{\theta})$ is not a 2-symmetric pair. The base space of the fibrations is associated with the 2-symmetric pair $(\mathfrak{g}_2, \mathfrak{a}_1 \oplus \mathfrak{a}_1)$ and hence it is the compact simply connected Cartan symmetric space given by $(\mathfrak{g}_{2(-14)}, \mathfrak{su}(2)+\mathfrak{su}(2))$. It has rank 2 and dimension 8. The fiber is associated with the 2-symmetric pair $(\mathfrak{a}_1 \oplus \mathfrak{a}_1, \mathfrak{a}_1 \oplus \mathfrak{T}^1)$. Here we have no problem in finding out which are the corresponding factors (because we are dealing with inner automorphisms). These are given by $(\mathfrak{a}_1, \mathfrak{a}_1) \times (\mathfrak{a}_1, \mathfrak{T}^1)$. The first factor is trivial and the second corresponds to $(\mathfrak{su}(2), \mathfrak{T}^1)$. The compact simply connected Cartan space is $SU(2)/SO(2) \cong SO(3)/SO(2) \cong S^2$. It is the two-dimensional sphere, and has rank one. There are two possibilities for the fiber, either it is $S^2$ or it is $\mathbb{R}\mathbb{P}^2$ the real projective plane. However, $\mathbb{R}\mathbb{P}^2$ is ruled out since it is not even orientable.

$\underline{\mathfrak{g} = \mathfrak{f}_4}$.

This simple Lie algebra has two involutions. Both of them are inner and their fixed point sets are given by $\mathfrak{b}_4$ and $\mathfrak{a}_1 \oplus \mathfrak{c}_3$. Now, the inner involutions of these two Lie algebras yield the following fixed point sets

| | | | |
|---|---|---|---|
| $\mathfrak{b}_4$ | $\mathfrak{b}_p \oplus \mathfrak{b}_{4-p}$ | $\mathfrak{a}_1 \oplus \mathfrak{c}_3$ | $\mathfrak{a}_1 \oplus (\mathfrak{c}_1 \oplus \mathfrak{c}_2)$ |
| | $2 \le p \le 4$ | | $\mathfrak{T}^1 \oplus (\mathfrak{c}_1 \oplus \mathfrak{c}_2)$ |
| | $\mathfrak{b}_3 \oplus \mathfrak{T}^1$ | | $\mathfrak{a}_1 \oplus (\mathfrak{a}_2 \oplus \mathfrak{T}^1)$ |
| | | | $\mathfrak{T}^1 \oplus (\mathfrak{a}_2 \oplus \mathfrak{T}^1)$ |
| | | | $\mathfrak{T}^1 \oplus \mathfrak{c}_3$ |

Thus we can draw the following table which also includes the case $\mathfrak{g}' = \mathfrak{g}_2$. As usual we refer to Table I for the Dynkin diagrams and notation.

| $\mathfrak{g}'$ | x | $\mathfrak{g}^{\theta}$ | $\mathfrak{g}^{\theta^2}$ | Base Space | Universal cover of the fiber |
|---|---|---|---|---|---|
| $\mathfrak{g}_2$ | $\frac{1}{2}\nu_2$ | $\mathfrak{a}_1 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_1$ | G | $S^2$ |
| | $\frac{3}{4}\nu_1$ | $\mathfrak{a}_1 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_1$ | G | $S^2$ |
| $\mathfrak{f}_4$ | $\frac{1}{2}\nu_1$ | $\mathfrak{c}_3 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{c}_3$ | FI | $S^2$ |
| | $\frac{1}{2}\nu_4$ | $\mathfrak{b}_3 \oplus \mathfrak{T}^1$ | $\mathfrak{b}_4$ | FII | $SO(9)/SO(7) \times SO(2)$ |
| | $\frac{3}{4}\nu_2$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{c}_3$ | FI | $Sp(3)/U(3)$ |
| | $\frac{1}{2}(\nu_1+\nu_4)$ | $\mathfrak{a}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{c}_3$ | FI | $S^2 \times (Sp(3)/Sp(1) \times Sp(2))$ |
| | $\nu_3$ | $\mathfrak{a}_3 \oplus \mathfrak{a}_1$ | $\mathfrak{b}_4$ | FII | $SO(9)/SO(6) \times SO(3)$ |

Here G. FI, FII stand for the corresponding compact simply connected Riemannian 2-symmetric spaces in Cartan's list. They are given by $(\mathfrak{g}_{2(-14)}, \mathfrak{su}(2) + \mathfrak{su}(2))$, $(\mathfrak{f}_{4(-52)}, \mathfrak{sp}(3) + \mathfrak{su}(2))$, $(\mathfrak{f}_{4(-52)}, \mathfrak{so}(9))$ respectively. Their dimensions are 8, 28, and 16 respectively, and their ranks 3, 4, 1 respectively.

We now draw tables for $\mathfrak{g} = \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$. We omit the details.

| | $\mathfrak{g}$ | $x$ | $\mathfrak{g}^{\theta}$ | $\mathfrak{g}^{\theta^2}$ | Base Space | Universal cover of the fiber |
|---|---|---|---|---|---|---|
| | $\mathfrak{e}_6$ | $\frac{1}{4}\nu_1$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^1$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^1$ | EIII | * |
| | | $\frac{1}{4}(\nu_1+\nu_6)$ | $\mathfrak{d}_4 \oplus \mathfrak{T}^2$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^1$ | EIII | SO(10)/SO(8) × SO(2) |
| | | $\frac{1}{2}\nu_2$ | $\mathfrak{a}_5 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_5 \oplus \mathfrak{a}_1$ | EII | $S^2$ |
| | | $\frac{1}{2}\nu_3$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_4 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_5 \oplus \mathfrak{a}_1$ | EII | $SU(6)/S(U_5 \times U_1)$ |
| ** | | $\frac{1}{4}(\nu_1+2\nu_3)$ | $\mathfrak{a}_4 \oplus \mathfrak{T}^2$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^1$ | EIII | SO(10)/U(5) |
| | | $\frac{1}{4}(\nu_1+2\nu_5)$ | $\mathfrak{a}_3 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^2$ | $\mathfrak{a}_5 \oplus \mathfrak{a}_1$ | EII | $S^2 \times (SU(6)/S(U_4 \times U_2))$ |
| ** | | $\frac{1}{4}(\nu_1+2\nu_2)$ | $\mathfrak{a}_4 \oplus \mathfrak{T}^2$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^1$ | EIII | SO(10)/U(5) |
| | | $\frac{3}{4}\nu_4$ | $\mathfrak{a}_2 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_5 \oplus \mathfrak{a}_1$ | EII | $SU(6)/S(U_3 \times U_3)$ |
| | | $\frac{1}{4}(2\nu_1+\nu_6)$ | $\mathfrak{d}_4 \oplus \mathfrak{T}^2$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^1$ | EIII | SO(10)/SO(8) × SO(2) |
| | | $\frac{1}{2}(\nu_3+\nu_5)$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_3 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^1$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^1$ | EIII | SO(10)/SO(4) × SO(6) |
| | | $\frac{1}{2}(\nu_2+\nu_5)$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_3 \oplus \mathfrak{T}^1$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^1$ | EIII | SO(10)/SO(4) × SO(6) |

<u>Remarks.</u> * Here the total space coincides with the base space. This means that the squares of the symmetries are the geodesic involutions and hence that the total space is an Hermitian 2-symmetric space of the given type.

** These are the only cases where some ambiguity arises. We have two possibilities for $\mathfrak{g}^{\theta^2}$ (however a direct computation in terms of root systems shows that $\mathfrak{a}_5 \oplus \mathfrak{a}_1$ is ruled out):

$\mathfrak{d}_5 \oplus \mathfrak{T}^1$ with fiber $SO(10)/U(5)$ or

$\mathfrak{a}_5 \oplus \mathfrak{a}_1$ with fiber $S^2 \times SU(6)/S(U_5 \times U_1)$.

EII stands for $(\mathfrak{e}_{6(-78)}, \mathfrak{su}(6) + \mathfrak{su}(2))$. It has rank 4 and dimension 40.

EIII stands for $(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbb{R})$. It has rank 2 and dimension 32.

| $\mathfrak{g}$ | x | $\mathfrak{g}^{\theta}$ | $\mathfrak{g}^{\theta^2}$ | Base Space | Universal cover of the fiber |
|---|---|---|---|---|---|
| $\mathfrak{e}_7$ | $\frac{1}{4}\nu_7$ | $\mathfrak{e}_6 \oplus \mathfrak{T}^1$ | $\mathfrak{e}_6 \oplus \mathfrak{T}^1$ | EVII | * |
| | $\frac{1}{2}\nu_1$ | $\mathfrak{d}_6 \oplus \mathfrak{T}^1$ | $\mathfrak{d}_6 \oplus \mathfrak{a}_1$ | EVI | $S^2$ |
| | $\frac{1}{2}\nu_6$ | $\mathfrak{d}_5 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{d}_6$ | EVI | $SO(12)/SO(10) \times SO(2)$ |
| | $\frac{1}{2}\nu_2$ | $\mathfrak{a}_6 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_7$ | EV | $SU(8)/S(U_7 \times U_1)$ |
| ** | $\frac{1}{4}(\nu_7+2\nu_1)$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^2$ | $\mathfrak{e}_6 \oplus \mathfrak{T}^1$ | EVII | EIII |
| ** | $\frac{1}{4}(\nu_7+2\nu_6)$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^2$ | $\mathfrak{e}_6 \oplus \mathfrak{T}^1$ | EVII | EIII |
| | $\frac{1}{4}(\nu_7+2\nu_2)$ | $\mathfrak{a}_5 \oplus \mathfrak{T}^2$ | $\mathfrak{a}_1 \oplus \mathfrak{d}_6$ | EVI | $S^2 \times (SO(12)/U(6))$ |
| | $\frac{3}{4}\nu_3$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_5 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{d}_6$ | EVI | $SO(12)/U(6)$ |
| | $\frac{3}{4}\nu_5$ | $\mathfrak{a}_4 \oplus \mathfrak{a}_2 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_7$ | EV | $SU(8)/S(U_5 \times U_3)$ |
| | $\frac{1}{2}(\nu_1+\nu_6)$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{d}_4 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{d}_6$ | EVI | $S^2 \times (SO(12)/SO(4) \times SO(8))$ |
| | $\frac{1}{2}(\nu_1+\nu_2)$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_5 \oplus \mathfrak{T}^1$ | $\mathfrak{e}_6 \oplus \mathfrak{T}^1$ | EVII | EII |
| | $\frac{1}{2}(\nu_2+\nu_6)$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_5 \oplus \mathfrak{T}^1$ | $\mathfrak{e}_6 \oplus \mathfrak{T}^1$ | EVII | EII |
| | $\nu_4$ | $\mathfrak{a}_3 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_3$ | $\mathfrak{a}_1 \oplus \mathfrak{d}_6$ | EVI | $SO(12)/SO(6) \times SO(6)$ |

Remarks. * Here the total space and the base space coincide. The squares of the symmetries are the geodesic involutions. This space is Hermitian 2-symmetric.

** In these entries some ambiguity arises since there are two possible candidates for $\mathfrak{g}^{\theta^2}$:

$\mathfrak{a}_1 \oplus \mathfrak{d}_6$ with fiber of type $S^2 \times (SO(12)/SO(10) \times SO(2))$

$\mathfrak{e}_6 \oplus \mathfrak{T}^1$ with fiber of type EIII.

However a direct computation in terms of root systems rules out the first alternative.

| $\mathfrak{g}$ | $x$ | $\mathfrak{g}^{\theta}$ | $\mathfrak{g}^{\theta^2}$ | Base Space | Universal cover of the fiber |
|---|---|---|---|---|---|
| $\mathfrak{e}_8$ | $\frac{1}{2}\nu_1$ | $\mathfrak{d}_7 \oplus \mathfrak{T}^1$ | $\mathfrak{d}_8$ | EVIII | $SO(16)/SO(14) \times SO(2)$ |
| | $\frac{1}{2}\nu_8$ | $\mathfrak{e}_7 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{e}_7$ | EIX | $S^2$ |
| | $\frac{3}{4}\nu_2$ | $\mathfrak{a}_7 \oplus \mathfrak{T}^1$ | $\mathfrak{d}_8$ | EVIII | $SO(16)/U(8)$ |
| | $\frac{3}{4}\nu_7$ | $\mathfrak{e}_6 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{e}_7$ | EIX | EVII |
| | $\frac{1}{2}(\nu_1 + \nu_8)$ | $\mathfrak{d}_6 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{e}_7$ | EIX | $S^2 \times$ (EVII) |
| | $\nu_3$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_7$ | $\mathfrak{a}_1 \oplus \mathfrak{e}_7$ | EIX | EV |
| | $\nu_6$ | $\mathfrak{d}_5 \oplus \mathfrak{a}_3$ | $\mathfrak{d}_8$ | EVIII | $SO(16)/SO(10) \times SO(6)$ |

For the classical Lie algebras one could follow the same procedure. However, it is possible to describe explicitly the automorphisms of order four by means of their standard matrix representations. We shall so this next. Note that a global formulation will be readily available.

Continued on p. 107
pp. 101 to 106 do not exist

## The Automorphisms of Order Four of the Classical Compact Lie Algebras

Here we describe the automorphisms of order four of the "classical" Lie algebras in much the same vein as Ch. X §2 in Helgason's book [9]. The main result is that we can obtain the compact simply connected 4-symmetric spaces associated with these Lie algebras. In fact, a complete description of the fibrations of these spaces is given. We list for each one of them both the fiber and the base space. Furthermore, the idea of duality for 2-symmetric spaces is extended to 4-symmetric spaces. This extension is very natural and provides us with a large class of examples of noncompact 4-symmetric spaces. A description of these dual spaces will also be given. Here we shall only consider inner automorphisms. For the outer automorphisms we refer to Section 8.(b).

Let $\mathfrak{u}$ be a compact simple Lie algebra and $\theta$ an automorphism of order four of $\mathfrak{u}$; let $\mathfrak{u} = \mathfrak{u}^{\theta} + \nu + h$ be the decomposition of $\mathfrak{u}$ into "eigenspaces" of $\theta$ (see Section 2 for details). Then $(\mathfrak{u}, \theta^2)$ is a 2-symmetric Lie algebra of the compact type and its dual Lie algebra $\mathfrak{g}_0 = (\mathfrak{u}^{\theta} + \nu) + \sqrt{-1}\, h$ is a 2-symmetric Lie algebra of the noncompact type. $\mathfrak{g}_0$ is a real form of the complexification $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$ and is also a 4-symmetric Lie algebra. It has the same fixed point set and the same "vertical space" as $\mathfrak{u}$. Hence the corresponding simply connected 4-symmetric space is Riemannian 4-symmetric with the same compact fiber as its dual. Note that duality does not go over the fibers.

Here we list the "classical $\mathfrak{u}$, that is, $\mathfrak{su}(n+1)$, $\mathfrak{so}(2n+1)$, $\mathfrak{so}(2n)$ and $\mathfrak{sp}(n)$ and for each give various $\theta$. We shall show that these $\theta$ exhaust all the possibilities up to conjugation. The simply connected Riemannian 4-symmetric spaces corresponding to $(\mathfrak{u}, \theta)$ and $\mathfrak{g}_0$ are also listed, along with the fiber and the base spaces.

The link between finite order automorphisms of a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and a compact real form $\mathfrak{u}$ (of $\mathfrak{g}$) is given by the following proposition. (The proof in [9] p.442 for the case $n = 2$ works in general). Here $\mathrm{Aut}(\mathfrak{g}, n)$ denotes the set of automorphisms of order $n$ for a Lie algebra $\mathfrak{g}$.

Proposition. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\mathfrak{u}$ a compact real form of $\mathfrak{g}$. The mapping

$$\tau : \mathrm{Aut}(\mathfrak{u}, n)/\mathrm{Aut}(\mathfrak{u}) \to \mathrm{Aut}(\mathfrak{g}, n)/\mathrm{Aut}(\mathfrak{g})$$

induced by $s \mapsto s^{\mathbb{C}}$ is a bijection. ///

For the definitions and notation of the classical Lie groups and their Lie algebras we refer to [9] Ch. X.

The unit matrix of order $n$ will be denoted by $I_n$, and we put

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad R_{p,q} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{p,q} & I_{p,q} \\ -I_{p,q} & I_{p,q} \end{pmatrix}$$

$$K_{p,q} = \begin{pmatrix} -I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ 0 & 0 & -I_p & 0 \\ 0 & 0 & 0 & I_q \end{pmatrix}, \quad L_{p,q} = \begin{pmatrix} 0 & 0 & I_p & 0 \\ 0 & I_q & 0 & 0 \\ -I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_q \end{pmatrix}$$

$$L_{p,q,r,s} = \begin{pmatrix} 0 & 0 & I_{p,q} & 0 \\ 0 & I_{r,s} & 0 & 0 \\ -I_{p,q} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{r,s} \end{pmatrix} \quad J_{p,q,r} = \begin{pmatrix} J_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & I_r \end{pmatrix}.$$

Remarks. $I_{q,p}$ and $J_{p,q,r}$ are orthogonal and have determinant $(-1)^q$. On the other hand, $J_n$, $R_{p,q}$, $K_{p,q}$, $L_{p,q,r,s}$ all belong to $SO(2n)$, $SU(2n)$ and $Sp(n)$ (for $n$ appropriate)

$\mathfrak{g} = \mathfrak{a}_n$, $n > 1$, $\mathfrak{u} = \mathfrak{su}(n+1)$.

The following proposition yields the classification of the inner automorphisms of order four of $\mathfrak{a}_n$.

Proposition. Let $\theta : \mathfrak{a}_n \to \mathfrak{a}_n$ $(n > 1)$ be an inner automorphism of order four. Then its fixed point set $\mathfrak{a}_n^\theta$ is one of the following.

(i) $\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{a}_{k-j-1} \oplus \mathfrak{a}_{n-k} \oplus \mathfrak{T}^3$, $1 \le i < j < k \le n$.

(ii) $\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{a}_{n-j} \oplus \mathfrak{T}^2$, $1 \le i < j \le n$.

(iii) $\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{n-i} \oplus \mathfrak{T}^1$, $1 \le i \le n$.

Conversely, each one of the above sets can be obtained as the fixed point set of an inner automorphism of order four of

$\mathfrak{a}_n$. Furthermore, the choice of this automorphism is unique up to conjugacy. ///

Corollary. Two automorphisms of $\mathfrak{a}_n$, $n > 1$ are conjugate if and only if their fixed point sets are isomorphic.

Proof. If both automorphisms are outer a glance at the classification table in Section 8(b) yields the result.

Remarks. In our classification tables (Table I) we have that for $\mathfrak{a}_n$ the automorphisms induced by $x = \frac{1}{4}(\nu_i + \nu_j)$ and $x = \frac{1}{4}(2\nu_i + \nu_j)$ $(i < j)$ both have the same fixed point set, hence they are conjugate (within the full group of automorphisms of $\mathfrak{a}_n$). For a proof of these results one has to apply the general theory as exposed in Helgason [9] Ch. X.

To construct the inner automorphisms of order four of $\mathfrak{su}(n+1)$, we regard this Lie algebra as imbedded in $\mathfrak{so}(2n+2)$ in the usual way, that is:

Let $X \in \mathfrak{su}(n+1)$, and write it as $A + iB$ with both $A$ and $B$ real, the imbedding is

$$A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathfrak{so}(2n+2).$$

The inner involutions of $\mathfrak{su}(n+1)$ (up to conjugation) are $X \mapsto I_{p,q} X I_{p,q}$ where $p + q = n + 1$, $p \geq q \geq 1$. When $\mathfrak{su}(n+1)$ is regarded as sitting inside $\mathfrak{so}(2n+2)$, these involutions are given by $X \mapsto K_{p,q} X K_{p,q}$, $X \in \mathfrak{su}(n+1) \subset \mathfrak{so}(2n+2)$. The fixed point sets of these involutions are $\mathfrak{su}(p) \oplus \mathfrak{t}_0 \oplus \mathfrak{su}(q)$

where $t_0$ is a one dimensional center. Note that in the complexification this set corresponds to $\mathfrak{a}_{p-1} \oplus \mathfrak{a}_{n-p} \oplus \mathfrak{T}^1$ $1 \leq p \leq n$, which is (iii) in the above list. This tells us that the corresponding 4-symmetric spaces are in fact Hermitian 2-symmetric with geodesic involutions the squares of the symmetries. These spaces are

$$SU(p,n+1-p)/S(U_p \times U_{n+1-p}), \quad SU(n+1)/S(U_p \times U_{n+1-p}) \quad (1 \leq p \leq n).$$

The automorphisms of order four on $\mathfrak{su}(n+1) \subset \mathfrak{so}(2n+2)$ are

$$X \mapsto L_{p,n+1-p} X \, L^{-1}_{p,n+1-p} \quad 1 \leq p \leq n.$$

(i) As it was pointed out, the matrix $L_{p,q,r,s}$ belongs to $SO(2(p+q+r+s))$ and hence. $X \mapsto L_{p,q,r,s} X \, L^{-1}_{p,q,r,s}$ defines an inner automorphism on $\mathfrak{so}(2n)$, $n = p + q + r + s$. The clue here is that this automorphism preserves $\mathfrak{su}(n)$. (This has to be proved).

Thus we let $\mathfrak{su}(n+1) \subset \mathfrak{so}(2n+2)$ and define $\theta$ by

$$X \mapsto L_{i,k-j,j-i,n-k+1} X \, L^{-1}_{i,k-j,j-i,n-k+1}, \quad (1 \leq i < j < k \leq m).$$

$\theta$ is an automorphism of order four whose fixed point set is

$$\mathfrak{u}^{\theta} = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \middle| \begin{array}{l} A \in \mathfrak{u}(i), \; B \in \mathfrak{u}(k-j), \; C \in \mathfrak{u}(j-i), \; D \in \mathfrak{u}(n-k+1) \\ Tr(A+B+C+D) = 0. \end{array} \right\}$$

Note: The description of $\mathfrak{u}^{\theta}$ is given regarding $\mathfrak{su}(n+1) \subset \mathfrak{gl}(n+1,\mathbb{C})$, a description of $\mathfrak{u}^{\theta}$ in $\mathfrak{so}(2n+2)$ would require an $8 \times 8$ pattern. Let

$$S(U_p \times U_q \times U_r \times U_s) = \left\{ \begin{pmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{pmatrix} \middle| \begin{array}{l} g_i \in U(i),\ i = p,q,r,s \\ \prod \det g_i = 1 \end{array} \right\}.$$

The corresponding simply connected 4-symmetric spaces are

$$SU(k-j+i, n-k+j-i+1)/S(U_i \times U_{k-j} \times U_{j-i} \times U_{n-k+1}),$$

$$SU(n+1)/S(U_i \times U_{k-j} \times U_{j-i} \times U_{n-k+1}), \quad (1 \le i < j < k \le n).$$

The fiber is

$$S(U_{k-j+i} \times U_{n-k+j-i+1})/S(U_i \times U_{k-j} \times U_{j-i} \times U_{n-k+1})$$

and the base spaces are

$$SU(k-j+i, n-k+j-i+1)/S(U_{k-j+i} \times U_{n-k+j-i+1})$$

$$SU(n+1)/S(U_{k-j+i} \times U_{n-k+j-i+1}).$$

(ii) This case is similar to (i). Again let $\mathfrak{su}(n+1) \subset \mathfrak{so}(2n+2)$ and define $\theta$ by

$$X \mapsto L_{i,n-j+1,0,j-i} X L^{-1}_{i,n-j+1,0,j-1} \quad 1 \le i < j \le n.$$

Note that this is the same as taking $i = 0$ or $k = n+1$ in (i).

The corresponding simply connected 4-symmetric spaces are

$$SU(n-j+i+1, j-i)/S(U_i \times U_{n-j+1} \times U_{j-i}),$$

$$SU(n+1)/S(U_i \times U_{n-j+1} \times U_{j-i}), \quad (1 \le i < j \le n)$$

with fiber

$$S(U_{n-j+i+1} \times U_{j-i})/S(U_i \times U_{n-j+1} \times U_{j-1})$$

and base spaces

$$SU(n-j+i+1, j-i)/S(U_{n-j+i+1} \times U_{j-i})$$

$$SU(n+1)/S(U_{n-j+i+1} \times U_{j-i}).$$

All this can be summarized as follows

Theorem. The compact simply connected 4-symmetric spaces with nonvanishing Euler characteristic associated with $\mathfrak{a}_n$, $n > 1$ are

$$SU(p+q+r+s)/S(U_p \times U_q \times U_r \times U_s) \quad p \geq 0,\ q \geq 1,\ r \geq 1,\ s \geq 0$$

with fiber

$$S(U_{p+q} \times U_{r+s})/S(U_p \times U_q \times U_r \times U_s)$$

and base

$$SU(p+q+r+s)/S(U_{p+q} \times U_{r+s}). \qquad ///$$

$\mathfrak{g} = \mathfrak{b}_n,\ n \geq 2,\ \mathfrak{n} = \mathfrak{so}(2n+1)$

Proposition. Let $\theta : \mathfrak{b}_n \to \mathfrak{b}_n$ be an inner automorphism of order four. Then its fixed point set $\mathfrak{b}_n^\theta$ is one of the following:

(i) $\mathfrak{a}_{i-2} \oplus \mathfrak{b}_{n-i} \oplus \mathfrak{T}^2$, $\quad 2 \leq i \leq n$

(ii) $\mathfrak{a}_{i-1} \oplus \mathfrak{b}_{n-i} \oplus \mathfrak{T}^1$, $\quad 2 \leq i \leq n$

(iii) $\mathfrak{d}_i \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{b}_{n-j} \oplus \mathfrak{T}^1$, $\quad 2 \leq i < j \leq n$

(iv) $\mathfrak{b}_{n-1} \oplus \mathfrak{T}^1$

Conversely, each one of the above sets can be obtained as the fixed point set of an inner automorphism of order four of $\mathfrak{b}_n$. Furthermore, the choice of this automorphism is unique up to conjugacy. ///

Corollary. Two automorphisms of $\mathfrak{b}_n$, $n > 1$ are conjugate if and only if their fixed point sets are isomorphic.

(i) Let $\theta$ be conjugation on $\mathfrak{so}(2n+1)$ with respect to $J_{i-1,2,2(n-i)+1}$ $(2 \le i \le n)$. The corresponding simply connected 4-symmetric spaces are

$$SO_0(2i-2,2(n-i)+3)/U(i-1) \times SO(2) \times SO(2(n-i)+1),$$

$$SO(2n+1)/U(i-1) \times SO(2) \times SO(2(n-i)+1), \quad (2 \le i \le n).$$

(ii) Let $\theta$ be conjugation on $\mathfrak{so}(2n+1)$ with respect to $J_{i,0,2(n-i)+1}$, $(2 \le i \le n)$. The corresponding simply connected 4-symmetric spaces are

$$SO_0(2i,2(n-i)+1)/U(i) \times SO(2(n-i)+1),$$

$$SO(2n+1)/U(i) \times SO(2(n-i)+1), \quad (2 \le i \le n).$$

(iii) Let $\theta$ be conjugation on $\mathfrak{so}(2n+1)$ with respect to $J_{j-i,2i,2(n-j)+1}$, $(2 \le i < j \le n)$. The corresponding simply connected 4-symmetric spaces are

$$SO_0(2(j-i),2(n-j+i)+1)/U(j-i) \times SO(2i) \times SO(2(n-j)+1),$$

$$SO(2n+1)/U(j-i) \times SO(2i) \times SO(2(n-j)+1) \quad 2 \le i < j \le n.$$

(iv) Let $\theta$ be conjugation on $\mathfrak{so}(2n+1)$ with respect to $J_{1,0}{}^{2n-1}$. The corresponding simply connected 4-symmetric spaces are

$$SO_0(2,2n-1)/U(1) \times SO(2n-1),\quad SO(2n+1)/SO(2) \times SO(2n-1).$$

These spaces are Hermitian 2-symmetric with geodesic involutions the squares of the symmetries.

In conclusion we have the following

<u>Theorem</u>. The compact simply connected 4-symmetric spaces associated with $\mathfrak{b}_n$, $(n \geq 2)$, are

$$SO(2n+1)/U(j-i) \times SO(2i) \times SO(2(n-j)+1) \qquad 0 \leq i < j \leq n$$

with base space

$$SO(2n+1)/SO(2(j-i)) \times SO(2(n-j+i)+1)$$

and fiber

$$(SO(2(j-i))/U(j-i)) \times (SO(2(n-j+i)+1)/SO(2i) \times SO(2(n-j)+1)). \text{ ///}$$

<u>Remarks</u>. For $\mathfrak{b}_n$ $(n \geq 2)$, there are no outer automorphisms of order four. Thus the above spaces are in fact all the possible compact simply connected 4-symmetric spaces associated with it. When $i = 0$ or $1$, we obtain the corresponding almost Hermitian 4-symmetric spaces. (See Section 9.)

<u>$\mathfrak{g} = \mathfrak{d}_n$, $(n > 3)$, $\mathfrak{u} = \mathfrak{so}(2n)$</u>

<u>Proposition</u>. Let $\theta : \mathfrak{d}_n \to \mathfrak{d}_n$, $n \geq 3$, be an inner automorphism

of order four. Then its fixed point set $\mathfrak{d}_n^\theta$ is one of the following

(i) $\mathfrak{a}_{n-3} \oplus \mathfrak{T}^3$

(ii) $\mathfrak{a}_{n-2} \oplus \mathfrak{T}^2$

(iii) (a) $\mathfrak{a}_{j-1} \oplus \mathfrak{a}_{n-j-1} \oplus \mathfrak{T}^2, \quad 2 \le j \le n-2$

(b) $\mathfrak{a}_{j-2} \oplus \mathfrak{d}_{n-j} \oplus \mathfrak{T}^2, \quad 2 \le j \le n-2$

(iv) $\mathfrak{a}_{j-1} \oplus \mathfrak{d}_{n-j} \oplus \mathfrak{T}^1, \quad 2 \le j \le n-2$

(v) $\mathfrak{d}_i \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{d}_{n-j} \oplus \mathfrak{T}^1, \quad 2 \le i < j \le n-2$

(vi) (a) $\mathfrak{a}_{n-1} \oplus \mathfrak{T}^1$

(b) $\mathfrak{d}_{n-1} \oplus \mathfrak{T}^1$.

Furthermore, for the case (ii) there are precisely two nonconjugate inner automorphisms of order four which have $\mathfrak{a}_{n-2} \oplus \mathfrak{T}^2$ as a fixed point set. For each remaining case, there correspond as many nonconjugate inner automorphisms of order four as many times as it appears in different series. ///

(i) I have not yet found a global formulation for this case, however a glance at the classification of the inner involutions of $\mathfrak{d}_n$, $n > 3$, yields the result that the corresponding base space must be $SO(2n)/SO(4) \times SO(2n-4)$, and that the universal cover of the fiber must be $S^2 \times S^2 \times SO(2n-4)/U(n-2)$.

(ii) We must find two nonconjugate automorphisms of order four with the same fixed point set.

Let $\theta_1 : \mathfrak{so}(2n) \to \mathfrak{so}(2n)$ be conjugation with respect to $R_{n-1,1}$, and let $\theta_2 : \mathfrak{so}(2n) \to \mathfrak{so}(2n)$ be conjugation with respect to $J_{n-1,0,2}$. Because $R^2_{n-1,1} = J_n$ and $J_{n-1,0,2} = I_{2n-2,2}$, they are nonconjugate. The corresponding simply connected 4-symmetric spaces are

$(\theta_1)$ $SO^*(2n/U(n-1) \times U(1)$, $SO(2n)/U(n-1) \times U(1)$

$(\theta_2)$ $SO_0(2n-2,2)/U(n-1) \times SO(2)$, $SO(2n)/U(n-1) \times SO(2)$.

At first there does not seem to be any difference in the compact case however, the corresponding fibrations are given as follows:

$$(\theta_1)\quad \begin{array}{c} U(n)/U(n-1)\times U(1) \hookrightarrow SO(2n)/U(n-1)\times U(1) \\ \downarrow \\ SO(2n)/U(n) \end{array} \qquad (\theta_2)\quad \begin{array}{c} SO(2n-2)/U(n-1) \hookrightarrow SO(2n)/U(n-1)\times SO(2) \\ \downarrow \\ SO(2n)/SO(2n-2)\times SO(2). \end{array}$$

(iii) (a) Let $\theta$ be conjugation on $\mathfrak{so}(2n)$ with respect to $R_{j,n-j}$, $(2 \le j \le n-2)$. The associated simply connected 4-symmetric spaces are

$SO^*(2n)/U(j)\times U(n-j)$, $SO(2n)/U(j)\times U(n-j)$, $(2 \le j \le n-2)$.

(b) Let $\theta$ be conjugation on $\mathfrak{so}(2n)$ with respect to $J_{j-1,2(n-j),2}$ $(2 \le j \le n-2)$. The corresponding simply connected 4-symmetric spaces are

$SO_0(2j-2,2(n-j)+2)/U(j-1) \times SO(2(n-j)) \times SO(2)$

$SO(2n)/U(j-1) \times SO(2(n-j)) \times SO(2)$, $(2 \le j \le n-2)$.

(iv) Let $\theta : \mathfrak{so}(2n) \to \mathfrak{so}(2n)$ be conjugation with respect to $J_{j,0,2(n-j)}$, $(2 \le j \le n-2)$. The corresponding simply connected 4-symmetric spaces are

$$SO_0(2j,2(n-j))/U(j)\times SO(2(n-j)),\ SO(2n)/U(j)\times SO(2(n-j)),\ (2\le j\le n-2).$$

(v) Let $\theta : \mathfrak{so}(2n) \to \mathfrak{so}(2n)$ be conjugation with respect to $J_{j-i,2i,2(n-j)}$, $(2 \le i < j \le n-2)$. The associated simply connected 4-symmetric spaces are

$$SO_0(2(j-i),2(n-j+1))/U(j-1) \times SO(2i) \times SO(2(n-j)),$$

$$SO(2n)/U(j-i)\times SO(2i)\times SO(2(n-j)),\ (2 \le i < j \le n-2).$$

(vi) (a) Let $\theta : \mathfrak{so}(2n) \to \mathfrak{so}(2n)$ be conjugation with respect to $R_{n,0}$. The corresponding simply connected 4-symmetric spaces are

$$SO^*(2n)/U(n),\ SO(2n)/U(n)$$

(b) Let $\theta : \mathfrak{so}(2n) \to \mathfrak{so}(2n)$ be conjugation with respect to $J_{1,0,2n-2}$. The corresponding simply connected 4-symmetric spaces are

$$SO_0(2,2n-2)/SO(2n-2) \times SO(2),\ SO(2n)/SO(2n-2) \times SO(2).$$

The spaces in (a) and in (b) in (vi) are Hermitian 2-symmetric with geodesic involutions the squares of the symmetries.

In conclusion we have

Theorem. The compact simply connected 4-symmetric spaces with non-vanishing Euler characteristic associated with $\mathfrak{d}_n$, $(n>3)$, are given by

| Total Space | Base Space | Fiber |
|---|---|---|
| $(\mathfrak{d}_n, \mathfrak{a}_{n-3} \oplus \mathfrak{T}^3)^*$ | | |
| $SO(2n)/U(n-1)\times U(1)$ | $SO(2n)/U(n)$ | $U(n)/U(n-1)\times U(1)$ |
| $SO(2n)/U(j)\times U(n-j)$<br>$1 \le j \le [\frac{1}{2}n]$ | $SO(2n)/$<br>$SO(2(n-j)) \times SO(2j)$ | $(SO(2j)/U(j))\times$<br>$(SO(2(n-j))/U(n-j))$ |
| $SO(2n)/U(j-i)\times$<br>$SO(2(n-j))\times SO(2i)$<br>$0 \le i < j \le n-2$ | $SO(2n)/$<br>$SO(2(j-i))\times SO(2(n-j+i))$ | $(SO(2(j-i))/U(j-i))\times$<br>$(SO(2(n-j+i))/SO(2i)\times$<br>$SO(2(n-j)))$ |
| $SO(2n)/U(n)$ | $SO(2n)/U(n)$ | |

<u>Remarks</u>. * Needs to be given a global formulation. The base space in this case is given by $SO(2n)/SO(4) \times SO(2n-4)$ and the universal cover of the fiber by $S^2 \times S^2 \times (SO(2n-4)/U(n-2))$.

<u>$\mathfrak{g} = \mathfrak{c}_n$, $(n > 2)$ $\mathfrak{k} = \mathfrak{sp}(n)$</u>

<u>Proposition</u>. Let $\theta : \mathfrak{c}_n \to \mathfrak{c}_n$ be an inner automorphism of order four. Then its fixed point set $\mathfrak{c}_n^\theta$ is given by one of the following

(i) $\mathfrak{a}_{n-1} \oplus \mathfrak{T}^1$

(ii) $\mathfrak{a}_{i-1} \oplus \mathfrak{c}_{n-i} \oplus \mathfrak{T}^1$, $\quad 1 \le i \le n-1$

(iii) $\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{n-i-1} \oplus \mathfrak{T}^2$, $\quad 1 \le i \le n-1$

(iv) $\mathfrak{c}_i \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{c}_{n-j} \oplus \mathfrak{T}^1$, $\quad 1 \le i < j \le n-1$

Conversely, each one of the above sets can be obtained as the fixed point set of an inner automorphism of order four of $\mathfrak{c}_n$. Furthermore, the choice of this automorphism is unique up to conjugacy. ///

Corollary. Two automorphisms of $\mathfrak{e}_n$ of order four, $(n > 2)$, are conjugate if and only if their fixed point sets are isomorphic. ///

We now construct the corresponding automorphisms for each fixed point set.

(i) Let $\theta : \mathfrak{sp}(n) \to \mathfrak{sp}(n)$ be conjugation with respect to $R_{0,n}$. The corresponding simply connected 4-symmetric spaces are

$$Sp(n,\mathbb{R})/U(n),\ Sp(n)/U(n).$$

These spaces are Hermitian 2-symmetric, the geodesic involutions are the squares of the symmetries, i.e. the vertical distribution is trivial.

(ii) Let $\theta : \mathfrak{sp}(n) \to \mathfrak{sp}(n)$ be conjugation with respect to $L_{i,n-i}$, $(1 \le i \le n-1)$. Here $\mathfrak{k}^{\theta}$ is given by $\mathfrak{k}(i) \oplus \mathfrak{sp}(n-i)$. The corresponding simply connected 4-symmetric spaces are

$$Sp(i,n-i)/U(i)\times Sp(n-i),\ Sp(n)/U(i)\times Sp(n-i),\ (1\le i\le n-1).$$

(iii) Let $\theta : \mathfrak{sp}(n) \to \mathfrak{sp}(n)$ be conjugation with respect to $R_{i,n-i}$, $(1 \le i \le n-1)$. The corresponding simply connected 4-symmetric spaces are

$$Sp(n,\mathbb{R})/U(i)\times U(n-i),\ Sp(n)/U(i)\times U(n-i),\ (1\le i\le n-1).$$

(iv) Let $\theta : \mathfrak{sp}(n) \to \mathfrak{sp}(n)$ be conjugation with respect to $L_{0,j-i,i,n-j}$, $(1 \le i < j \le n-1)$. Here the fixed point set is given by $\mathfrak{k}(j-i) \oplus \mathfrak{sp}(i) \oplus \mathfrak{sp}(n-j)$. The corresponding simply connected 4-symmetric spaces are

$Sp(j-i, n-j+i)/U(j-i)\times Sp(i)\times Sp(n-j)$,

$Sp(n)/U(j-i)\times Sp(i)\times Sp(n-j)$, $1 \le i < j \le n-1$.

In conclusion we have the following

Theorem. The compact simply conencted 4-symmetric spaces associated with $\mathfrak{C}_n$, $(n \ge 3)$, are

| Total Space | Base Space | Fiber |
|---|---|---|
| $Sp(n)/U(i)\times U(n-i)$ <br> $0 \le i \le n-1$ | $Sp(n)/U(n)$ | $U(n)/U(i)\times U(n-i)$ |
| $Sp(n)/U(j-i)\times$ <br> $Sp(i)\times Sp(n-j)$ <br> $0 \le i < j \le n-1$ | $Sp(n)/(Sp(j-i)\times$ <br> $Sp(n-j+i))$ | $(Sp(j-i)/U(j-i))\times$ <br> $(Sp(n-j+i)/Sp(n-j)\times Sp(i))$ |

Remarks. $\mathfrak{C}_n$ has no outer automorphisms of order four. Hence the above spaces provide all the compact simply connected 4-symmetric spaces associated with $\mathfrak{C}_n$ $(n \ge 3)$. The spaces appearing in the first row, and those in the second row with $i = 0$ are the almost Hermitian 4-symmetric spaces associated with $\mathfrak{C}_n (n \ge 3)$.

## (b) Outer Automorphisms of Order Four of Compact Simple Lie Algebras

As for inner automorphisms, we first reduce the problem of classification of the pair $(\mathfrak{g},\sigma)$ with $\mathfrak{g}$ compact semi-simple and $\sigma$ automorphism of order four, to the case when $\mathfrak{g}$ is simple. The difference is that if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ is the decomposition of $\mathfrak{g}$ into its simple ideals, then $\sigma$ does not necessarily preserve it. However it can at most permute the factors. Let $\{\mathfrak{g}_i, \ldots, \mathfrak{g}_r\}$ be an orbit by $\sigma$. Since $\sigma$ is of order four, it will contain at most four elements. We assume the indices are such that $\sigma(\mathfrak{g}_i) = \mathfrak{g}_{i+1}$, and so on, with $\sigma(\mathfrak{g}_r) = \mathfrak{g}_i$. Then $\sigma$ restricted to $\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_r$ is an automorphism of order four with each one of the ideals isomorphic to $\mathfrak{g}_i$. Thus we may think of $\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_r$ as

$\underbrace{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_i}_{r-i \text{ times}}$ and of $\sigma$ as the automorphism $(x_i,\ldots,x_r) \mapsto (\sigma(x_r),x_i,x_{i+1},\ldots,x_{r-1})$. $\sigma$ is outer if either the orbit $\mathfrak{g}_i,\ldots,\mathfrak{g}_r$ contains more than one element or if $\sigma : \mathfrak{g}_r \to \mathfrak{g}_i$ is outer. Hence we can restrict our attention to outer automorphisms of order four of compact simple Lie algebras.

The way we obtain the classification of the outer automorphisms of order four of the simple Lie algebras over $\mathbb{C}$ is by a direct application of the general method exposed in [ 9 ] Ch. X. We will not give the details of this method. We shall only enunciate the main theorems and then proceed to the classification. At the end of the section, we give explicit descriptions of the fibrations of the spaces. Once again the global formulation is presented for $\mathfrak{g}$ "classical".

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, then there exists a one to one correspondence between automorphisms $\sigma$ of $\mathfrak{g}$ of order $m$ and $\mathbb{Z}_m$-gradations of $\mathfrak{g}$. $\mathfrak{g} = \oplus_{i\in\mathbb{Z}_m}\mathfrak{g}_i$. Associated with each one of these gradations is the covering Lie algebra $L(\mathfrak{g},\sigma) = \oplus_{j\in\mathbb{Z}} x^j \mathfrak{g}_{j \bmod m}$ and the covering homomorphism $\phi : L(\mathfrak{g},\sigma) \to \mathfrak{g}$ defined by $\phi(x^k Y) = Y$ $(Y \in \mathfrak{g}_{k \bmod m})$. The main point is that by studying the $\mathbb{Z}$-graded Lie algebras $L(\mathfrak{g},\sigma)$ ($\mathfrak{g}$ simple) one obtains a description of all $\mathbb{Z}_m$-grad ations of simple Lie algebras. The idea is to develop the weight theory for $L(\mathfrak{g},\sigma)$, that is, the analog of the root theory for $\mathfrak{g}$.

Thereby one establishes an isomorphism $L(\mathfrak{g},\sigma) \approx L(\mathfrak{g},\nu)$, where $\nu$ is an automorphism of a very special type, namely induced by an automorphism of the Dynkin diagram. This results in an explicit description of $\sigma$ in terms of $\nu$.

The diagrams associated with these covering Lie algebras are given in the table in p. 503 in [ 9 ]. For the notion of canonical set of generators we refer to the same book, page 482. We now state the main theorem.

Theorem. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, $\nu$ a fixed automorphism of $\mathfrak{g}$ of order $k$ $(k = 1,2,3)$ induced by an automorphism of the Dynkin diagram for a Cartan subalgebra $\tilde{h}$ of $\mathfrak{g}$. Let $\mathfrak{g} = \oplus_i \mathfrak{g}_i^{\nu}$ be the corresponding $\mathbf{Z}_k$-gradation. The fixed point set $h^{\nu}$ of $\nu$ in $\tilde{h}$ is a Cartan subalgebra of the (simple) Lie algebra $\mathfrak{g}_0^{\nu}$. Fix canonical generators $X_i$, $Y_i$, $H_i$ $(1 \le i \le n)$ of $\mathfrak{g}_0^{\nu}$ corresponding to the simple roots $\alpha_1,\ldots,\alpha_n$ in $\Delta(\mathfrak{g}_0^{\nu}, h^{\nu})$. Let $\tilde{\alpha}_0$ be the lowest root of $L(\mathfrak{g},\nu)$ of the form $(\alpha_0,1)$ and fix $X_0 \neq 0$ in $\mathfrak{g}_1^{\nu}$ such that $xX_0 \in L(\mathfrak{g},\nu)^{\tilde{\alpha}_0}$. Let $(s_0,\ldots,s_n)$ be integers $\geq 0$ without nontrivial common factor. Put $m = k \sum_0^n a_i s_i$ where the $a_i$ are the labels from the diagram of $L(\mathfrak{g},\nu)$ corresponding to the simple roots $\tilde{\alpha}_0$, $\tilde{\alpha}_i = (\alpha_i,0)$, $(1 \le i \le n)$. Let $\varepsilon$ be a fixed $m$-th root of unity. Then:

(i) The vectors $X_0, X_1,\ldots,X_n$ generate $\mathfrak{g}$ and the relations

$$\sigma(X_i) = \varepsilon^{s_i} X_i \quad (0 \le i \le n)$$

define uniquely an automorphism of $\mathfrak{g}$ of order $m$. It will be called an automorphism of type $(s_0,\ldots,s_n;k)$.

(ii) Let $i_1,\ldots,i_t$ be all the indices for which $s_{i_1}=\ldots=s_{i_t}=0$. Then $\mathfrak{g}_0$ (that is $\mathfrak{g}_0^\sigma$) is the direct sum of an $(n-t)$-dimensional center and a semisimple Lie algebra whose Dynkin diagram is the subdiagram of the diagram $\mathfrak{g}^{(k)}$ in Table k consisting of the vertices $i_1,\ldots,i_t$. (See table in [9], p. 503).

(iii) Except for conjugation, the automorphisms $\sigma$ exhaust all n-th order automorphisms of $\mathfrak{g}$.

<u>Theorem</u>. With the notation of the above theorem, let $\sigma$ be an automorphism of type $(s_0,\ldots,s_n;k)$. Then

(i) $\sigma$ is an inner automorphism if and only if $k = 1$.

(ii) If $\sigma'$ is an automorphism of type $(s_0',\ldots,s_n';k')$, then $\sigma$ and $\sigma'$ are conjugate within $\mathrm{Aut}(\mathfrak{g})$ if and only if $k = k'$ and the sequence $(s_0,\ldots,s_n)$ can be transformed into the sequence $(s_0',\ldots,s_n')$ by an automorphism $\psi_0$ of the diagram $\mathfrak{g}^{(k)}$.

We now classify the outer automorphisms of order 4 up to conjugacy. The equation to solve is $4 = k\sum_{i=0}^{n} a_i s_i$. However, since we are interested in outer automorphisms, the second theorem tells us that $k = 2$. The following are the solutions.

(i) $a_i = a_j = s_i = s_j = 1$.

(ii) $a_i = 2$, $s_i = 1$.

In case (i) $\mathfrak{g}^{\sigma}$ has one dimensional center, whereas in case (ii), $\mathfrak{g}^{\theta}$ is semisimple.

Outer Automorphism of Order Four of the Simple Lie Algebras

| $\mathfrak{g}$ | $\mathfrak{g}^{\sigma}$ |
|---|---|
| $\mathfrak{a}_2$ | $\mathfrak{a}_1$ |
| $\mathfrak{a}_{2n}$<br>$n>1$ | $\mathfrak{c}_j\oplus\mathfrak{b}_{n-j}$<br>$1\le j\le n$ |
| $\mathfrak{a}_{2n-1}$<br>$n>2$ | $\mathfrak{d}_i\oplus\mathfrak{c}_{n-i}$<br>$2\le i\le n-1$ |
| $\mathfrak{e}_6$ | $\mathfrak{a}_3\oplus\mathfrak{a}_1$ |
| | $\mathfrak{b}_3\oplus\mathfrak{a}_1$ |

| $\mathfrak{g}$ | $\mathfrak{g}^{\sigma}$ |
|---|---|
| $\mathfrak{a}_{2n-1}$<br>$n>2$ | $\mathfrak{c}_{n-1}\oplus\mathfrak{T}^1$ |
| | $\mathfrak{a}_{n-1}\oplus\mathfrak{T}^1$ |
| $\mathfrak{d}_{n+1}$<br>$n>1$ | $\mathfrak{b}_{i-1}\oplus\mathfrak{a}_{n-i-j}\oplus\mathfrak{b}_j\oplus\mathfrak{T}^1$<br>$1\le i\le n,\ 0\le j\le n-i$ |
| $\mathfrak{e}_6$ | $\mathfrak{c}_3\oplus\mathfrak{T}^1$ |

Remark. For $\mathfrak{d}_{n+1}$ there does exist symmetry for the diagram and hence some restriction $0 \le i \le [\frac{1}{2}(n+1)]$ has to be made to avoid repetition. Also, any two automorphisms giving rise to the same entry $(\mathfrak{g},\mathfrak{g}^{\sigma})$ are conjugate.

We now set about finding $\mathfrak{g}^{\sigma^2}$. We follow the method as explained in part (a) of this section. We start with $\mathfrak{e}_6$ and $\mathfrak{a}_{2n}, n > 1$. These cases are straightforward. We omit the details. For notation we refer to Helgason p. 518.

| $\mathfrak{g}$ | $\mathfrak{g}^{\sigma}$ | $\mathfrak{g}^{\sigma^2}$ | Base space | Universal cover of the fiber |
|---|---|---|---|---|
| $\mathfrak{e}_6$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_3$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_5$ | EII | $SU(6)/SO(6)$ |
| | $\mathfrak{b}_3 \oplus \mathfrak{a}_1$ | $\mathfrak{d}_5 \oplus \mathfrak{T}^1$ | EIII | $(SO(10)/SO(7)\times SO(3)) \times \mathbb{R}$ |
| | $\mathfrak{c}_3 \oplus \mathfrak{T}^1$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_5$ | EII | $S^2 \times (SU(6)/Sp(3))$ |
| $\mathfrak{a}_{2n}$ $n>1$ | $\mathfrak{c}_j \oplus \mathfrak{b}_{n-j}$ $1\le j\le n$ | $\mathfrak{a}_{2j-1} \oplus \mathfrak{a}_{2(n-j)} \oplus \mathfrak{T}^1$ | $SU(2n+1)/S(U_{2j}\times U_{2(n-j)+1})$ | $(SU(2j)/Sp(j)) \times (SU(2(n-j)+1)/SO(2(n-j)+1)) \times \mathbb{R}$ |

Remark. EII and EIII stand for the compact simply connected Riemannian 2-symmetric spaces in Cartan's classification list (see Helgason pp. 517, 518).

By considering $\mathfrak{a}_{2n}$ as the Lie algebra of complex $(2n+1) \times (2n+1)$ matrices of trace $0$, we can give an explicit construction of its outer automorphisms of order four. Note that all we need to do is to construct outer automorphisms of order four whose fixed point sets are as given in the table. Then recall that any two such automorphisms giving rise to the same entry $(\mathfrak{g}, \mathfrak{g}^{\sigma})$ are conjugate. Therefore, these will in fact be all the possibilities.

One further comment. It turns out that these automorphisms leave invariant the compact real form $\mathfrak{su}(2n+1)$. Thus we would rather work with this Lie algebra. As a consequence, at the end we obtain a global formulation. That is, we obtain the compact simply connected 4-symmetric spaces associated with the entries $(\mathfrak{a}_{2n}, \mathfrak{a}_{2n}^{\sigma})$ $\sigma$ outer of order four.

The following notation will be useful (see Helgason [ 9 ] Ch. X, §2). $I_n$ will denote the unit matrix of order $n$, and put

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

For $0 \le i \le n-1$ define $\sum_i = \begin{pmatrix} J_{n-i} & 0 \\ 0 & I_{2i+1} \end{pmatrix}$, and define $\sigma_i : \mathfrak{su}(2n+1) \to \mathfrak{su}(2n+1)$ as the automorphism $\sigma_i(X) = \sum_i \overline{X} \sum_i^{-1}$. Here the bar denotes conjugation in the usual sense.

<u>Claim</u>. $\sigma_i$ is of order four and has fixed point set given by $\mathfrak{sp}(n-i) \times \mathfrak{so}(2i+1)$. (I.e. complexifying $\mathfrak{sp}(n-i,\mathbb{C}) \oplus \mathfrak{so}(2i+1,\mathbb{C}) = \mathfrak{c}_{n-i} \oplus \mathfrak{b}_i$). Furthermore, $\sigma_i^2(X) = I_{2(n-i),2i+1}$ has fixed point set $\mathfrak{su}(2(n-i)) \times \mathfrak{t}_0 \times \mathfrak{su}(2i+1)$ (here $\mathfrak{t}_0$ denotes a one-dimensional center). The proof of this fact can safely be omitted.

Thus $\sigma_i$, $0 \le i \le n-1$ are all the outer automorphisms of order four (up to conjugation). Denoting by the same letter the induced automorphisms on $SU(2n+1)$, we have that the corresponding compact simply connected 4-symmetric spaces are given by $SU(2n+1)/Sp(n-i) \times SO(2i+1)$.

The base space is given by $SU(2n+1)/S(U_{2(n-i)} \times U_{2i+1})$, and the fiber by $S(U_{2(n-i)} \times U_{2i+1})/Sp(n-i) \times SO(2i+1)$.

Schematically, we have

$$\begin{tikzcd} S(U_{2(n-i)} \times U_{2i+1})/Sp(n-i)\times SO(2i+1) \arrow[r, hook] & SU(2n+1)/Sp(n-i)\times SO(2i+1) \arrow[d] \\ & SU(2n+1)/S(U_{2(n-i)} \times U_{2i+1}). \end{tikzcd}$$

We can do the same thing for $\mathfrak{a}_{2n-1}$ $n > 2$. Now instead of $\Sigma_i$ and $\sigma_i$, define $R_i = \begin{pmatrix} J_{n-i} & 0 \\ 0 & I_{2i} \end{pmatrix}$ $1 \leq i \leq n-1$ and $\rho_i(X) = R_i \bar{X} R_i^{-1}$.

Claim. $\rho_i (2 \leq i \leq n-1)$ is an automorphism of order four with fixed point set $\mathfrak{sp}(n-i) \oplus \mathfrak{so}(2i)$ (i.e. complexifying $\mathfrak{c}_{n-i} \oplus \mathfrak{d}_i$ $2 \leq i \leq n-1$). Furthermore $\rho_i^2 = I_{2(n-i),2i}$ has fixed point set $\mathfrak{su}(2(n-i)) \times t_0 \times \mathfrak{su}(2i)$. ($t_0$ is a one-dimensional center). The corresponding fibrations are given by

$$\begin{tikzcd} S(U_{2(n-i)} \times U_{2i})/Sp(n-i)\times SO(2i) \arrow[r, hook] & SU(2n)/Sp(n-i)\times SO(2i) \arrow[d] \\ & SU(2n)/S(U_{2(n-i)} \times U_{2i}). \end{tikzcd}$$

On the other hand we have $\rho_1 : \mathfrak{su}(2n) \to \mathfrak{su}(2n)$ with fixed point set $\mathfrak{sp}(n-1) \oplus \mathfrak{so}(2)$ when complexified becomes $\mathfrak{c}_{n-1} \oplus \mathfrak{T}^1$ one dimensional center. The corresponding fibration is given by

$$\begin{tikzcd} S(U_{2n-2} \times U_2)/Sp(n-1)\times SO(2) \arrow[r, hook] & SU(2n)/Sp(n-1)\times SO(2) \arrow[d] \\ & SU(2n)/S(U_{2n-2} \times U_2). \end{tikzcd}$$

This is the only case where the fiber and hence the total space appear to admit invariant almost complex structures. However, this is not so. Note that the base is always an Hermitian

2-symmetric space in the three cases.

For $\mathfrak{a}_{2n-1}$ $(n > 1)$ there is still one more outer automorphism of order four. This automorphism must have as its fixed point set $\mathfrak{a}_{n-1} \oplus \mathfrak{T}^1$. However, I have not yet found an explicit description as in the above cases.

We now consider $\mathfrak{g} = \mathfrak{d}_{n+1}$, $(n > 1)$. The Lie algebra of complex skew symmetric matrices of order $2(n+1)$ and take its compact real form $\mathfrak{so}(2n+2)$.

Let $2 \le i + j \le n - 1$, $1 \le i \le n - 1$, $0 \le j$. Define

$$\Sigma_{ij} = \begin{pmatrix} J_{n+1-i-j} & 0 \\ 0 & I_{2j+1,2i-1} \end{pmatrix} \quad \text{and} \quad \begin{aligned} \sigma_{ij} : \mathfrak{so}(2n+2) &\to \mathfrak{so}(2n+2) \\ X &\mapsto \Sigma_{ij} X \Sigma_{ij}^{-1}. \end{aligned}$$

<u>Claim</u>. $\sigma_{ij}$ is of order four and its fixed point set is given by $\mathfrak{u}(n-i-j+1) \oplus \mathfrak{so}(2j+1) \oplus \mathfrak{so}(2i-1)$. (I.e. when complexified $\mathfrak{a}_{n-i-j} \oplus \mathfrak{T}^1 \oplus \mathfrak{b}_j \oplus \mathfrak{b}_{i-1}$). The fixed point set of $\sigma_{ij}^2 = I_{2(n+1-i-j),2(i+j)}$ is given by $\mathfrak{so}(2(n+1-i-j)) \times \mathfrak{so}(2(i+j))$. ///

Since conjugation by $\Sigma_{ij}$ also induces an automorphism of $SO(2n+2)$, we have that the corresponding fibrations are given by

$$\begin{array}{c} (SO(2(n+1-i-j))/U(n+1-i-j)) \times (SO(2(i+j))/SO(2j+1) \times SO(2i-1)) \hookrightarrow \\ SO(2n+2)/U(n+1-i-j) \times SO(2j+1) \times SO(2i-1) \\ \downarrow \\ SO(2n+2)/SO(2(n+1-i-j)) \times SO(2(i+j)). \end{array}$$

We now consider (in the above situation) the excluded possibility $i+j=n$. In this case we have

$$ {}_n\Sigma_{ij} = \begin{pmatrix} J_1 & 0 \\ 0 & I_{2j+1,2i-1} \end{pmatrix} \text{ and } \begin{aligned} {}_n\sigma_{ij} : \mathfrak{so}(2n+2) &\to \mathfrak{so}(2n+2) \\ X &\mapsto {}_n\Sigma_{ij} X \, {}_n\Sigma_{ij}^{-1}. \end{aligned} $$

(We put the index $n$ to remember that $i+j=n$).

<u>Claim</u>. ${}_n\sigma_{ij}$ is an automorphism of order four whose fixed point set is given by $\mathfrak{so}(2) \times \mathfrak{so}(2j+1) \times \mathfrak{so}(2i-1)$. (In the complexification it is given by $\mathfrak{T}^1 \oplus \mathfrak{b}_j \oplus \mathfrak{b}_{i-1}$ $i+j=n$, $1 \leq i \leq n$). ${}_n\sigma_{ij}^2 = I_{2,2n}$ has fixed point set $\mathfrak{so}(2) \oplus \mathfrak{so}(2n)$. ///

Once again we obtain an induced automorphism on $SO(2n+2)$, and hence, we obtain the corresponding fibrations

$$ \begin{array}{ccc} SO(2n)/SO(2(n-i)+1)\times SO(2i-1) & \hookrightarrow & SO(2n+2)/SO(2)\times SO(2(n-i+1)\times SO(2i-1) \\ & & \downarrow \\ & & SO(2n+2)/SO(2)\times SO(2n). \end{array} $$

There is one outer automorphism of order four for $\mathfrak{d}_{n+1}$, which we ought to determine, this automorphism has fixed point set $\mathfrak{a}_{n-1} \oplus \mathfrak{T}^1$. (It corresponds to $i=1$, $j=0$ in our classification tables). For this, take conjugation on $\mathfrak{so}(2n+2)$ with respect to the matrix $\begin{pmatrix} J_n & 0 \\ 0 & \begin{pmatrix} 0,1 \\ 1,0 \end{pmatrix} \end{pmatrix}$. The fixed point set is $\mathfrak{u}(n)$. The corresponding fibration is

$$ \begin{array}{ccc} SO(2) \times (SO(2n)/U(n)) & \hookrightarrow & SO(2n+2)/U(n) \\ & & \downarrow \\ & & SO(2n+2)/SO(2) \times SO(2n). \end{array} $$

For the entry $(\mathfrak{a}_2', \mathfrak{a}_1)$ one can do a direct computation. One obtains:

$SU(2) \hookrightarrow U(2) \hookrightarrow SU(3)$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b \\ 0 & \overline{\det g} & 0 \\ c & 0 & d \end{pmatrix}$$

$$\begin{array}{c} U(2)/SU(2) \hookrightarrow SU(3)/SU(2) \\ \downarrow \\ SU(3)/U(2) \end{array}$$

The dual spaces are given as follows.

| | |
|---|---|
| $SU(2n+1)/Sp(n-i)\times SO(2i+1)$<br>$0 \leq i \leq n-1,\ n > 1$ | $SU(2(n-i),2i+1)/Sp(n-i)\times SO(2i+1)$ |
| $SU(2n)/Sp(n-i)\times SO(2i)$<br>$1 \leq i \leq n-1,\ n > 2$ | $SU(2(n-i),2i)/Sp(n-i)\times SO(2i)$ |
| $SU(3)/SU(2)$ | $SU(2,1)/SU(2)$ |
| $SO(2n+2)/U(n+1-i-j)\times$<br>$SO(2j+1)\times SO(2i-1)$<br>$2 \leq i+j \leq n-1,$<br>$1 \leq i \leq n-1,\ 0 \leq j,\ n > 1$ | $SO_0(2(n+1-i-j),2(i+j))/$<br>$U(n+1-i-j)\times SO(2j+1)\times SO(2i-1)$ |
| $SO(2n+2)/SO(2)\times$<br>$SO(2(n-i)+1)\times SO(2i-1)$<br>$1 \leq i \leq n,\ n > 1$ | $SO_0(2,2n)/SO(2)\times SO(2(n-i)+1)\times$<br>$SO(2i-1)$ |
| $SO(2n+2)/U(n)$ | $SO_0(2,2n)/U(n)$ |

The following table is a summary of our above discussion, it gives the description of the fibrations for the compact simply connected 4-symmetric spaces with vanishing characteristic associated with the "classical" Lie algebras.

G classical, rank G > rank K

| Total space | Base space | Fiber |
|---|---|---|
| $SU(2n+1)/Sp(n-i)\times SO(2i+1)$<br>$0\le i\le n-1,\ n>1$ | $SU(2n+1)/S(U_{2(n-i)}\times U_{2i+1})$ | $S(U_{2(n-i)}\times U_{2i+1})/Sp(n-i)\times SO(2i+1)$ |
| $SU(2n)/Sp(n-i)\times SO(2i)$<br>$1\le i\le n-1,\ n>2$ | $SU(2n)/S(U_{2(n-i)}\times U_{2i})$ | $S(U_{2(n-i)}\times U_{2i})/Sp(n-i)\times SO(2i)$ |
| * $(\mathfrak{a}_{2n-1}, \mathfrak{a}_{n-1}\oplus\mathfrak{T}^1)$ | | |
| $SU(3)/SU(2)$ | $SU(3)/U(2)$ | $U(2)/SU(2)$ |
| $SO(2n+2)/U(n+1-i-j)\times SO(2j+1)\times SO(2i-1)$<br>$2\le i+j\le n-1,\ 1\le i\le n-1,\ 0\le j,\ n>1$ | $SO(2n+2)/SO(2(n+1-i-j))\times SO(2(i+j))$ | $(SO(2(n+1-i-j))/U(n+1-i-j))\ \times$<br>$(SO(2(i+j))/SO(2j+1)\times SO(2i-1))$ |
| $SO(2n+2)/SO(2)\times SO(2(n-i)+1)\times SO(2i-1)$<br>$i+j=n,\ 1\le i\le n,\ n>1$ | $SO(2n+2)/SO(2)\times SO(2n)$ | $SO(2n)/SO(2(n-i)+1)\times SO(2i-1)$ |
| $SO(2n+2)/U(n)$ | $SO(2n+2)/SO(2)\times SO(2n)$ | $SO(2)\times(SO(2n)/U(n))$ |

Remarks. * A global formulation has to be obtained.

## §9. Almost Hermitian 4-Symmetric Spaces of Positive Characteristic

An almost Hermitian 4-symmetric space is a Riemannian 4-symmetric space which has an almost complex structure invariant under each symmetry and which is compatible with the Riemannian metric. Examples are provided by the Hermitian 2-symmetric spaces of the compact type and the noncompact type. These are always simply connected and have the characteristic property that their isotropy groups are not semisimple and therefore have nondiscrete centers. Actually, these spaces are Hermitian $n$-symmetric for any $n$ (see [27]). Here we shall be concerned with compact almost Hermitian 4-symmetric spaces with nonvanishing Euler characteristic. We show that these spaces are characterized as the homogeneous manifolds of the form $G/K$, where $G$ is a compact connected Lie group acting effectively on $G/K$, and $K$ is a connected (closed) subgroup of maximal rank with the following two properties. (i) It is the identity component of the fixed pointset of an (inner)automorphism $\theta$ of $G$ of order four inducing the symmetries on $M$, and (ii) it is the centralizer of a torus. We then classify them.

This section was motivated by the following two results for Riemannian 2-symmetric spaces: (i) Every almost complex structure invariant under the geodesic symmetries is integrable and (ii) if in addition this structure is compatible with the Riemannian metric, then it is Kählerian. Hence, it is natural to ask whether or not an invariant almost complex structure on a Riemannian 4-symmetric space is integrable or if compatible with the metric is Kählerian. From our representation of almost Hermitian 4-symmetric spaces with nonvanishing characteristic it will turn out that they admit invariant complex structures which are not only Kähler but also Hodge. In particular they are algebraic manifolds.

First of all, we solve a more basic problem. This is the question of existence of such almost complex structures.

(Q) Decide which of the compact Riemannian 4-symmetric spaces with nonvanishing Euler characteristic admit an invariant almost complex structure.

The way we proceed to answer (Q) is as follows: First we prove that these spaces have a decomposition into "simple factors" (for details see below) and that each of these simple factors also belongs to this same class of spaces. The simple factors are quotients of compact simple Lie group Now we can use all the information contained in the classification tables of Section 8. From these tables, we distinguish between two possibilities for $\mathfrak{k}$ (the Lie algebra of fixed points of the automorphism $\theta$): (i) $\mathfrak{k}$ is the the centralizer of a torus, (ii) $\mathfrak{k}$ is not the centralizer of a torus. In the first case it is immediate that the corresponding space admits an invariant almost complex structure. For the second case, we draw tables of these spaces. The final result being that only those spaces falling into the first class do admit invariant almost complex structures.

It should be pointed out that in a more general setting, Wolf and Gray (see [28]) have already given a complete description of all almost complex manifolds $(M,J)$, $M = G/K$, where $G$ is a compact connected Lie group acting effectively on $M$, $K$ is a subgroup of maximal rank (this is an equivalent way of saying that $G/K$ has nonvanishing Euler characteristic), and $J$ is a $G$-invariant almost complex structure on $M$.

We could reformulate our original question (Q) as follows: from the description given by Wolf and Gray find those spaces which are Riemannian 4-symmetric.

First we recall a decomposition that holds in general for compact homogeneous spaces with nonvanishing Euler characteristic. The details may be found for example in [24]. Let $M = G/K$ be a homogeneous space where $G$ is a compact connected Lie group acting effectively on $M$ and $K$ is a connected subgroup of maximal rank. As $K$ contains the center of $G$, this means that $G$ is semisimple and centerless. Now

$$(1) \quad G = G_1 \times \dots \times G_r, \quad K = K_1 \times \dots \times K_r, \quad M = M_1 \times \dots \times M_r$$

where

$$G_i \text{ is simple}, \quad K_i = K \cap G_i, \quad M_i = G_i/K_i.$$

Following Wolf and Gray, we shall refer to (1) as the decomposition of $M = G/K$ into simple factors. The point is that if $M = G/K$ is assumed to be an almost-Hermitian 4-symmetric space, then its simple factors are also almost-Hermitian 4-symmetric spaces. This is the content of the next theorem.

Before stating the theorem, we shall make a comment on the notation that we shall be using throughout this section: It has already been proved in Section 2 that every Riemannian 4-symmetric space can be represented as an homogeneous space of the form $M = G/K$, where $G$ is a connected Lie group of isometries acting effectively on $M$ which is invariant under conjugation by the symmetries and $K$, the isotropy group at a point $0 \in M$ say, is an open subgroup of the fixed point set $G^\theta$ where $\theta$ is the automorphism on $G$ induced by conjugation by the symmetry at $0$. We shall refer to such a representation as a standard representation. Whenever a Riemannian 4-symmetric space is written as coset space, it will be understood that it is by means of a standard representation.

Let $(M,g,J)$ be an almost Hermitian 4-symmetric space. Then the set of isometries of $M$ that preserves the almost complex structure is a closed subgroup of $I(M,g)$ and is therefore a Lie transformation group of $M$. We denote this group by $A(M)$. It is transitive on $M$ since it contains all the symmetries. The identity component $A_0(M)$ of $A(M)$ is also transitive on $M$. Let $0 \in M$ and let $K$ be the subgroup of $G = A_0(M)$ leaving $0$ fixed. With the automorphism $g \to s_0 g s_0^{-1}$ of $G$ (denoted by $\theta$), $M = G/K$ is a standard representation of $M$ with the additional property that the almost complex structure $J$ of $M$ is $G$-invariant. In particular, if $M$ is compact with nonvanishing Euler class, then $M$ has a standard representation with the following properties:

(a) $M = G/K$, $G$ compact connected Lie group of isometries acting effectively on $M$ which also preserves the almost complex structure $J$.

(b) $K$ is a closed connected subgroup of maximal rank.

(c) The automorphism $\theta$ of $G$ is an inner automorphism of $G$. It actually can be written as $\theta = \mathrm{Ad}(k)$ for some $k \in K$.

In particular, these spaces are simply connected. The proofs of (b) and (c) can be found in §4 of [28].

Theorem 1. Let $(M,g,J)$ be a compact almost Hermitian 4-symmetric space with nonvanishing Euler characteristic. Let $M = G/K$ be a standard representation with properties (a), (b) and (c) as above. Then $M = G/K$ decomposes into prime factors as in (1), i.e.

$$G = G_1 \times \ldots \times G_r \quad K = K_1 \times \ldots \times K_r, \quad M = M_1 \times \ldots \times M_r$$

where

$$G_i \text{ is simple,} \qquad K_i = K \cap G_i \qquad M_i = G_i/K_i$$

Furthermore, each of the prime factors $M_i = G_i/K_i$ is also a compact almost Hermitian 4-symmetric space with nonvanishing Euler class, and the representation $G_i/K_i$ is standard and satisfies properties (a), (b), (c) as well.

Proof. First we quote a result which gives us the induced Riemannian metrics and the induced almost complex structures on the prime factors.

Proposition 1 (see §4, [28]). Let $T$ be the class of tensor fields of one of the following types: complex structure, almost complex structure, Riemannian metric, almost Hermitian metric or Hermitian metric. Then the G-invariant tensor fields of type $T$ on $M = G/K$ are just the tensor fields $\xi = \xi_1 \oplus \ldots \oplus \xi_r$ where in the above notation, $\xi_i$ is an arbitrary $G_i$-invariant tensor field of type $T$ on $M_i$.

The proof of the theorem is now very straightforward. The automorphism of $G$ is inner, therefore it decomposes into a product $\theta = \theta_1 \times \ldots \times \theta_r$ of inner automorphisms of the factors $G_i$. Clearly $K_i$ is left fixed by $\theta_i$ which implies that $G_i/K_i$ is a standard representation of $M_i$. $\theta_i = \mathrm{Ad}(k_i)$ for some $k_i \in K_i$ (by (c)) and hence the symmetries preserve the metric and the almost complex structure.

Thus question (Q) can be reformulated as follows:

(Q') Let $(M,g,J)$ be a compact almost-Hermitian 4-symmetric space. Assume that $M$ admits a standard representation $M = G/K$ with properties (a), (b) and (c), and such that $G$ is a simple Lie group.

Problem: classify these spaces.

The rest of this section is devoted to solving this problem. This will be done from the point of view of Lie algebras.

First, since we are dealing with inner automorphisms, we distinguish from the tables in §8.(a) when $\mathfrak{k}$ is or is not the centralizer of a torus. For convenience we quote the criterion that gives us necessary and sufficient conditions that the fixed point set of an inner automorphism be the centralizer of a torus. (See [28] Prop. 2.11).

Proposition 2. Let $\theta = \mathrm{Ad}(\exp 2\pi\sqrt{-1}\, x)$ be an inner automorphism of a compact simple Lie algebra $\mathfrak{g}$ with maximal root $\mu = m_i\psi_i$, in a simple root system $\psi = \{\psi_1,\ldots,\psi_\ell\}$. Normalize $x = \sum_{i=1}^{\ell} c_i v_i$ as in Section 8(a). Then

1. $\mathfrak{g}^\theta$ is the centralizer of a torus if and only if one of the following two conditions holds true

(1.a) $\mu(x) < 1$;

(1.b) $\mu(x) = 1$, $c_i > 0$ implies that $m_i > 1$, and $\{m_i : c_i > 0\}$ is a set of $r \geq 2$ relatively prime integers.

2. $\mathfrak{g}^\theta$ is not the centralizer of a torus if and only if the following three conditions are satisfied

($\alpha$) $\mu(x) = 1$,

($\beta$) $c_i > 0$ implies $m_i > 1$,

($\gamma$) $\{m_i : c_i > 0\}$ either has just one element or is a set of $r \geq 2$ integers with greatest common divisor $p = 2$, 3, or 4.

In the former case of (iii), $x = \nu_j$ and $\theta$ has order $m_j > 1$; in the latter case of (iii) if $\theta$ has order $k$ then $p$ divides $k$.

A direct application of this criterion to the description given in Proposition 2 in Section 8.(a) of the inner automorphisms of order four of the compact simple Lie algebras, yields the result that for the first three cases there (i), (ii) and (iii), the corresponding fixed point set $\mathfrak{g}^{\theta}$ of $\theta = \mathrm{Ad}(\exp 2\pi\sqrt{-1}\, x)$ is the centrl izer of a torus. Whereas for the forth case $\mathfrak{g}^{\theta}$ is not the centralizer of a torus.

From the classification tables in Section 8.(a) it is now easy to draw the following table which gives a complete list of the possibilities for $x$ an element in $\sqrt{-1}\,\mathfrak{t}_0$ inducing an automorphism of order four $\theta = \mathrm{Ad}(\exp 2\pi\sqrt{-1}\, x)$ such that its fixed point set $\mathfrak{g}^{\theta}$ is not the centralizer of a torus. $\psi_x$ is a simple root system of $\mathfrak{g}^{\theta}$. We refer the reader to the tables in Section 8.(a) for notation and the Dynkin diagrams.

| $\mathfrak{g}$ | $x$ | $\psi_x$ | $\mathfrak{g}^{\theta}$ |
|---|---|---|---|
| $\mathfrak{b}_n, n\geq 2$ | $\frac{1}{2}(\nu_i+\nu_j)(2\leq i<j\leq n)$ | $\{\alpha_1,\ldots,\alpha_{i-1};\alpha_{i+1},\ldots,\alpha_{j-1};$ $\alpha_{j+1},\ldots,\alpha_n\}\cup\{-\mu\}$ | $\mathfrak{b}_i\oplus\mathfrak{a}_{j-i-1}$ $\oplus\mathfrak{b}_{n-j}\oplus\mathfrak{T}^1$ |
| $\mathfrak{c}_n, n\geq 3$ | $\frac{1}{2}(\nu_i+\nu_j)(1\leq i<j\leq n-1)$ | $\{\alpha_1,\ldots,\alpha_{i-1};\alpha_{i+1},\ldots,\alpha_{j-1};$ $\alpha_{j+1},\ldots,\alpha_n\}\cup\{-\mu\}$ | $\mathfrak{c}_i\oplus\mathfrak{a}_{j-i-1}$ $\oplus\mathfrak{c}_{n-j}\oplus\mathfrak{T}^1$ |

| $\mathfrak{g}$ | $x$ | $\psi_x$ | $\mathfrak{g}^\theta$ |
|---|---|---|---|
| $\mathfrak{d}_n, n\geq 4$ | $\frac{1}{2}(\nu_i+\nu_j)(2\leq i<j\leq n-2)$ | $\{\alpha_1,\ldots,\alpha_{i-1};\alpha_{i+1},\ldots,\alpha_{j-1};$ $\alpha_{j+1},\ldots,\alpha_n\}\cup\{-\mu\}$ | $\mathfrak{d}_i \oplus \mathfrak{a}_{j-i-1}$ $\oplus \mathfrak{d}_{n-j} \oplus \mathfrak{T}^1$ |
| $\mathfrak{f}_4$ | $\frac{1}{2}(\nu_1+\nu_4)$ | $\{\alpha_2,\alpha_3\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus {}_2 \oplus \mathfrak{T}^1$ |
| | $\nu_3$ | $\{\alpha_1,\alpha_2;\alpha_4\}\cup\{-\mu\}$ | $\mathfrak{a}_3 \oplus \mathfrak{a}_1$ |
| $\mathfrak{e}_6$ | $\frac{1}{2}(\nu_3+\nu_5)$ | $\{\alpha_1,\alpha_2,\alpha_4,\alpha_6\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_3 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}(\nu_2+\nu_3)$ $[\sim\frac{1}{2}(\nu_2+\nu_5)]$ | $\{\alpha_1,\alpha_4,\alpha_5,\alpha_6\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_3 \oplus \mathfrak{T}^1$ |
| $\mathfrak{e}_7$ | $\frac{1}{2}(\nu_1+\nu_6)$ | $\{\alpha_2,\ldots,\alpha_5;\alpha_7\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{d}_4 \oplus \mathfrak{T}^1$ |
| | $\frac{1}{2}(\nu_1+\nu_2)$ | $\{\alpha_3,\ldots,\alpha_7\}\cup\{-\mu\}$ | $\mathfrak{a}_5 \oplus \mathfrak{T}^1 \oplus \mathfrak{a}_1$ |
| | $\frac{1}{2}(\nu_2+\nu_6)$ | $\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_7\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_5 \oplus \mathfrak{T}^1$ |
| | $\nu_4$ | $\{\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6,\alpha_7\}\cup\{-\mu\}$ | $\mathfrak{a}_3 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_3$ |
| $\mathfrak{e}_8$ | $\frac{1}{2}(\nu_1+\nu_8)$ | $\{\alpha_2,\ldots,\alpha_7\}\cup\{-\mu\}$ | $\mathfrak{d}_6 \oplus \mathfrak{a}_1 \oplus \mathfrak{T}^1$ |
| | $\nu_3$ | $\{\alpha_1;\alpha_2,\alpha_4,\ldots,\alpha_8\}\cup\{-\mu\}$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_7$ |
| | $\nu_6$ | $\{\alpha_1,\ldots,\alpha_5;\alpha_7,\alpha_8\}\cup\{-\mu\}$ | $\mathfrak{d}_5 \oplus \mathfrak{a}_3$ |

The problem now is to decide whether or not each of the spaces in the above table admits an invariant almost complex structure. Wolf and Gray have classified the spaces $(G/K,J)$ where $G$ is a compact connected simple Lie group, $K$ is a connected subgroup of maximal rank which is not the centralizer of a torus and $J$ is an invariant almost complex structure on $G/K$ [28]. Thus a case by case inspection shows that none of the spaces in the table admits an invariant almost complex structure. Hence we have proved the following

<u>Theorem 2</u>. Let $(M,g,J)$ be a compact almost Hermitian 4-symmetric space with nonvanishing Euler characteristic. Let $M = G/K$ be a standard representation with properties (a), (b) and (c). Assume further that $G$ is simple. Then $M = G/K$ is one of the spaces listed in Section 8.(a) excluding the spaces that appear in the above table.

Theorems 1 and 2 yield the classification of compact almost Hermitian 4-symmetric spaces with nonvanishing Euler characteristic. The following theorem gives a characterization of these spaces.

<u>Theorem 3</u>. Let $(M,g)$ be a compact Riemannian 4-symmetric space with nonvanishing Euler characteristic. $M$ admits an invariant almost complex structure if and only if $M$ has a standard representation $M = G/K$ where $K$ is the centralizer of a torus.

Having answered question (Q) we now turn to study the geometry of these spaces. One thing that is worth pointing out is that the classification was obtained mainly by using the symmetries (homogeneity) and the invariant almost complex structure. No allusion was made to any kind of

Riemannian metric. We shall now start doing this. This is only a beginning and in no way constitutes an exhaustive treatment of the subject. We shall answer the original questions that motivated this section, through the relation between 3- and 4-symmetric spaces.

($Q_1$) Decide whether or not an invariant almost complex structure on a Riemannian 4-symmetric space is integrable.

($Q_2$) Decide whether or not an almost Hermitian 4-symmetric space is a Kähler manifold.

Using our characterization in Theorem 3, ($Q_1$) and ($Q_2$) have the following (partial) answers.

Theorem 4. Let $(M,g,J)$ be a compact almost Hermitian 4-symmetric space with nonvanishing Euler characteristic. Then $M$ admits an invariant almost Hermitian metric which makes it into a Hodge manifold. In particular, $M$ is an algebraic manifold.

Remarks. This theorem does not affirm anything about the original almost Hermitian structure, it only ensures the existence of such a structure with the stated properties. For a proof see [28], Section 9.

On the other hand, we prove that every almost Hermitian 4-symmetric space admits invariant almost Hermitian structure which is non-Kählerian, hence solving ($Q_2$).

Theorem 5. Let $M$ be an almost Hermitian 4-symmetric space. Assume that the vertical distribution $V$ has positive dimension. Then $M$ admits an invariant almost Hermitian metric which is non-Kählerian.

Comments. The condition that $V$ be nontrivial is necessary; otherwise the squares of the symmetries would be the usual geodesic involutions and hence the space would be Hermitian 2-symmetric and in particular Kähler. Examples of spaces where this happens are provided by those corresponding to the entries where $x = \frac{1}{4} \nu_i$ with $m_i = 1$ in the notation of Section 8.(a), (Proposition 3, and the tables).

Before we prove the theorem, we give a criterium which tells us when a Riemannian 4-symmetric space is an almost Hermitian 4-symmetric space. This theorem points in a different direction to Theorem 2. First, the nonvanishing condition on the Euler characteristic is dropped, second, it holds true in general whether or not the space be compact or noncompact and third, no mention is made to any transitive group of isometries. As an application, we shall use it at the end of the section to decide for the majority of cases, when the compact simply connected space (with vanishing Euler characteristic) corresponding to one of the entries in the classification in Section 8.(b) is an almost Hermitian 4-symmetric space.

Theorem 6. Let $(M,g)$ be a Riemannian 4-symmetric space. Then $(M,g)$ is an almost Hermitian 4-symmetric space if and only if the fiber of $M$ is an Hermitian 2-symmetric space.

Proof. Assume $M$ is an almost Hermitian 4-symmetric space, and let $\tilde{J}$ denote the invariant almost complex structure on $M$.

Claim. $\tilde{J}$ preserves both the vertical distribution $V$ and the horizontal distribution $H$. The proof can safely be omitted. Note that $SJ = JS$ where $S$ is the symmetry tensor.

Thus we have that the restriction of $J$ to the fibers defines an almost complex structure on them. These fibers are Cartan symmetric space, and since the geodesic involutions are induced by the symmetries (of $M$), they leave invariant the almost complex structure. Hence the fibers are Hermitian 2-symmetric.

For the converse, let $\bar{J}$ be the complex structure on the fibers, then we only have to define an invariant almost complex structure on $H$. But this is provided by $S$ when restricted to $H$. Define $J = \bar{J}$ on $V$ and $J = S$ on $H$. ///

Proof of Theorem 5. The idea is very simple. We use the almost complex structure $J$ on $M$ constructed above and observe that if it is Kählerian, then a new structure $\tilde{J}$ can be defined which is non-Kählerian.

In general, for an almost Hermitian 4-symmetric space we have $(\nabla_V J)W = 0$, $V(\nabla_H J)V = 0$ and $H(\nabla_H J)K = 0$. Thus, the space is Kählerian if and only if the following three conditions are satisfied:

$$(\nabla_V J)H = 0, \quad \mathcal{H}(\nabla_H J)V = 0 \quad \text{and} \quad \mathcal{V}(\nabla_H J)K = 0.$$

Now observe that the second and third conditions establish a relation between $\bar{J}$ and $S$:

$$\mathcal{H}(\nabla_H J)V = \mathcal{H}\nabla_H(\bar{J}V) - S\mathcal{H}\nabla_H V = 0 \quad \text{and} \quad \mathcal{V}(\nabla_H J)K = \mathcal{V}\nabla_H(SK) - \bar{J}\mathcal{V}\nabla_H K.$$

Hence if $J$ is Kählerian, the new almost complex structure $\tilde{J}$ defined as follows

$\tilde{J} = -\bar{J}$ on $\mathcal{V}$, $J = S$ on $\mathcal{H}$ will not be Kählerian. ///

Invariant almost complex structures on compact 4-symmetric spaces with vanishing Euler characteristic.

A glance at the classification table in Section 8(b) shows that with possibly the exception of the space associated with $(\mathfrak{a}_{2n-1}, \mathfrak{a}_{n-1} \oplus \mathfrak{T}^1)$, none of the spaces admits an invariant almost complex structure.

Remark. The difficulty in deciding whether or not the spaces associated to $(\mathfrak{a}_{2n-1}, \mathfrak{a}_{n-1} \oplus \mathfrak{T}^1)$ are Hermitian 4-symmetric springs from the fact that the fixed point set of $\sigma^2$ does not have an immediate description. (Here $\sigma$ is the corresponding automorphism of order four of $\mathfrak{a}_{2n-1}$). However, the author conjectures that this space is not almost Hermitian 4-symmetric.

We state the following conjecture.

Conjecture. Let $(M,g)$ be a compact simply connected almost Hermitian 4-symmetric space. Then $M$ has nonvanishing Euler characteristic.

The relation between 3- and 4-symmetric spaces.

We have already pointed out that the Hermitian 2-symmetric spaces are Hermitian $n$-symmetric for any order $n$. On the other hand, since J. Wolf and A. Gray [28] have already classified the 3-symmetric pairs $(G,K)$ with $G$ compact simple, it is also interesting to see which of these pairs will also correspond to 4-symmetric pairs. We shall see that all the pairs which are "good candidates" are in fact 4-symmetric. As an application of this relationship, we shall be able to give a final answer to question $(Q_1)$ in the negative sense, (that is, not all invariant almost complex structures are integrable).

First we describe what we mean by a "good candidate". Consider a 3-symmetric pair $(G,K)$. (We shall always assume $G$ compact and simple). Then either $\operatorname{rank} K = \operatorname{rank} G$ or $\operatorname{rank} K < \operatorname{rank} G$. The second case, is immediately ruled out by classification. Thus we are left with the case of equal rank. That is $G/K$ must have nonvanishing Euler characteristic. We now resort to the canonical almost complex structure of a 3-symmetric space. This structure is $G$-invariant, thus, in particular, if the pair $(G,K)$ is 4-symmetric, it will be invariant under the symmetries of order four (because they necessarily are inner). Hence $G/K$ would have to be an almost-Hermitian 4-symmetric space. Then Theorem 3 above, tells us that in this case $K$ has to be the centralizer of a torus. This tells us that a compact 3-symmetric pair $(G,K)$ ($G$ simple) is a "good candidate" to be a 4-symmetric pair only if $K$ is the centralizer of a torus. We shall prove that this condition is sufficient.

Theorem. Let $(G,K)$ be a 3-symmetric pair with $G$ compact simple Lie group. Assume further that rank $G$ = rank $K$ and that $K$ is the centralizer of a torus. Then $(G,K)$ is a 4-symmetric pair.

Remark. If $(G,K)$ is a pair as in the theorem, then $K$ is connected and $G/K$ is compact simply connected. Hence, the problem can be transformed into a problem on Lie algebra. Thus we prove:

Theorem. Let $(\mathfrak{g},\mathfrak{k})$ be a 3-symmetric pair with $\mathfrak{g}$ compact simple Lie algebra. Assume further that rank $\mathfrak{k}$ = rank $\mathfrak{g}$ and that $\mathfrak{k}$ is the centralizer of a toral subalgebra. Then $(\mathfrak{g},\mathfrak{k})$ is a 4-symmetric pair.

The following proposition is the analog of Proposition 2 in Section 8(a). It gives the local classification of 3-symmetric spaces - which is what we need. For notation we refer to Section 8(a).

Proposition ([28]p. 87). Let $\varphi$ be an inner automorphism of order 3 on a compact or complex simple Lie algebra $\mathfrak{g}$.

Choose a Cartan subalgebra and let $\psi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a simple root system for $\mathfrak{g}$. Then $\varphi$ is conjugate in the inner automorphism group of $\mathfrak{g}$ to some $\theta = \mathrm{Ad}(\exp 2\pi\sqrt{-1}\, x)$ where either $x = \frac{1}{3} m_i \nu_i$ with $1 \le m_i \le 3$ or $x = \frac{1}{3}(\nu_i + \nu_j)$ with $m_i = m_j = 1$.

<u>Remark</u>. A complete list of the possibilities for $x$, the fixed point set $\mathfrak{g}^\theta$ and a simple root system $\psi_x$ of $\mathfrak{g}^\theta$, up to conjugacy in the automorphism group of $\mathfrak{g}$, can be found in the tables in [28] pp. 88-89.

The next step in our program is to decide for which of the $x$'s as described in the proposition, $\mathfrak{g}^\theta$ is the centralizer of a toral subalgebra. But for this we can apply the criterion given in Proposition 2 in this section. This yields the following conclusions.

(*) If $x = \frac{1}{3} m_i \nu_i$ with $1 \le m_i \le 2$ or $x = \frac{1}{3}(\nu_i + \nu_j)$ with $m_i = m_j = 1$, then $\mathfrak{g}^\theta$ is the centralizer of a toral subalgebra.

(**) If $x = \frac{1}{3} m_i \nu_i$ with $m_i = 3$, $\mathfrak{g}^\theta$ is not the centralizer of a toral subalgebra. Thus we restrict our attention to (*).

If $x = \frac{1}{3} m_i \nu_i$ with $m_i = 1,2$. Then $\mathfrak{g}^\theta = \mathfrak{g}^\sigma$ with $\sigma = \mathrm{Ad}(\exp 2\pi\sqrt{-1}\, \frac{1}{4} m_i \nu_i)$ and $\sigma$ is of order four. Analogously for $x = \frac{1}{3}(\nu_i + \nu_j)$ with $m_i = m_j = 1$, we have that $\mathfrak{g}^\theta = \mathfrak{g}^\sigma$ with $\sigma = \mathrm{Ad}(\exp 2\pi\sqrt{-1}\, \frac{1}{4}(\nu_i + \nu_j))$ and $\sigma$ is of order four. This completes the proofs of the theorems.

We now apply this to answer $(Q_1)$ (and hence $(Q_2)$). A. Gray [8] showed that every 3-symmetric space admits a nearly Kähler structure. Furthermore, he showed that the structure is Kählerian if and only if the space is Hermitian 2-symmetric ([8] p. 353). On the other hand, in [15] it is proved that $M = SU(3)/T^2$ is not homeomorphic with the underlying manifold of any Riemannian 2-symmetric space. Since $M$ is a 3-symmetric space, the above two results say that its nearly Kähler structure cannot be Kählerian. Since the almost complex structure of a non-Kähler nearly Kähler manifold is never integrable, and since the pair $(SU(3), T^2)$ is also 4-symmetric, we obtain an example of an invariant almost complex structure on a 4-symmetric (and also on a 3-symmetric) space which is not integrable.

BIBLIOGRAPHY

[1] Ambrose, W., Parallel translation of Riemannian curvature. Ann. of Math. 64(1956), 337-363.

[2] Ambrose, W. and Singer, I. M., On homogeneous Riemannian manifolds. Duke Math. J. 25 (1958), 647-669.

[3] Chavel, I., "Riemannian Symmetric Spaces of Rank One." Dekkar, New York, 1972.

[4] Chevalley, C., "Theory of Lie groups", Vol. I Princeton Univ. Press, Princeton, N. J. 1946.

[5] Escobales, Jr., R. H., Riemannian submersions with totally geodesic fibers, J. Diff. Geom. 10 (1975) 253-276.

[6] Escobales, Jr., R. H., Riemannian submersions from complex projective space. J. Diff. Geom. 13 (1978) 93-107.

[7] Graham, P. J. and Ledger, A. J., s -Regular Manifolds, Diff. Geom. in honor of K. Yano, Kinoluniga, Tokyo, 1972, 133-144. (See also [30]).

[8] Gray, A., Riemannian manifolds with geodesic symmetries of order 3. J. Diff. Geom. (1972) 343-369.

[9] Helgason, S., "Differential Geometry, Lie Groups and Symmetric Spaces," Academic Press, 1978.

[10] Hermann, R., Compact homogeneous almost complex spaces of positive characteristic. Trans. Amer. Math. Soc. Vol. 83 (1956) 471-481.

[11] Johnson, D. L., Kähler submersions and holomorphic connections. J. Diff. Geom. Vol. 15 (1980) 71-79.

[12] Kobayashi, S. and Nomizu, K., "Foundations of Differential Geometry I, II," Wiley (Interscience) New York 1963 and 1969.

[13] Kowalsky, O., Riemannian manifolds with general symmetries. Math. Z. 136 (1974) 137-150.

[14] Kowalsky, O., Existence of generalized symmetric Riemannian spaces of arbitrary order. J. Diff. Geom Vol. 12, No. 2 (1977) 203-208.

[15] Ledger, A. J. and Pettit, R. B., Classification of metrizable regular s-manifolds with integrable symmetry tensor field. J. Math. Soc. of Japan. Vol. 28, No. 4 (1976) 668-675. (See also [29]).

[16] Ledger, A. J. and Obata, M., Affine and Riemannian s-manifolds. J. Diff. Geom. 2 (1968), 451-459.

[17] Loos, O. Spiegelungsräume und homogene symmetrische Räume. Math. 2. 99 (1967) 141-170.

[18] Loos, O., Reflection spaces of Minimal and Maximal Torsion. Math. Zeitschr. 106 (1968) 67-72.

[19] Milnor, J., "Morse Theory." Ann. of Math. Studies, Vol. 15, Princeton, Univ. Press, Princeton, N.J. 1963.

[20] O'Neill, B., The fundamental equations of a submersion. Michigan Math. J. Vol. 13 (1966) 459-469.

[21] O'Neill, B., Submersions and Geodesics. Duke Math. J. 34 (1967) 363-373.

[22] Palais, R. S., A global formulation of the Lie theory of transformation groups. Memoirs Amer. Math. Soc. No. 27, 1957.

[23] Steenrod, N. "Topology of Fiber Bundles." Princeton Univ. Press, Princeton, N.J. 1951.

[24] Wang, H. C., Homogeneous spaces with non-vanishing Euler characteristics. Ann. Math. Vol. 50 (1949) 925-953.

[25] Wang, H. C., Closed manifolds with a homogeneous complex structure. Amer. J. Math. 76 (1954) 1-32.

[26] Warner, F., "Foundations of Differentiable Manifolds and Lie Groups," Scott Foresman, Glenview, Illinois, 1970.

[27] Wolf, J. A., "Spaces of Constant Curvature," 4th Edition 1977 Publish or Perish.

[28] Wolf, J. A. and Gray, A., Homogeneous spaces defined by Lie group automorphisms I, II. J. Diff. Geom. 2 (1968) 77-159.

[29] Ledger, A. J., E spaces de Riemann symétriques généralisés, C.R.Acad. Sc. Paris, 246 (1967) 947-948.

[30] Graham, P.J. and Ledger A.J., Sur une classe de s-variétés Riemanniennes ou affines, C.R. Acad. Sci. Paris 267 (1968) 105-107.

[31] Kowalski, O. Generalized Symmetric Spaces, Springer-Verlag, Berlin, Heidelberg, New York, 1980.